Introduction to Effective-one-body (EOB) formalism and gravitational waveform construction

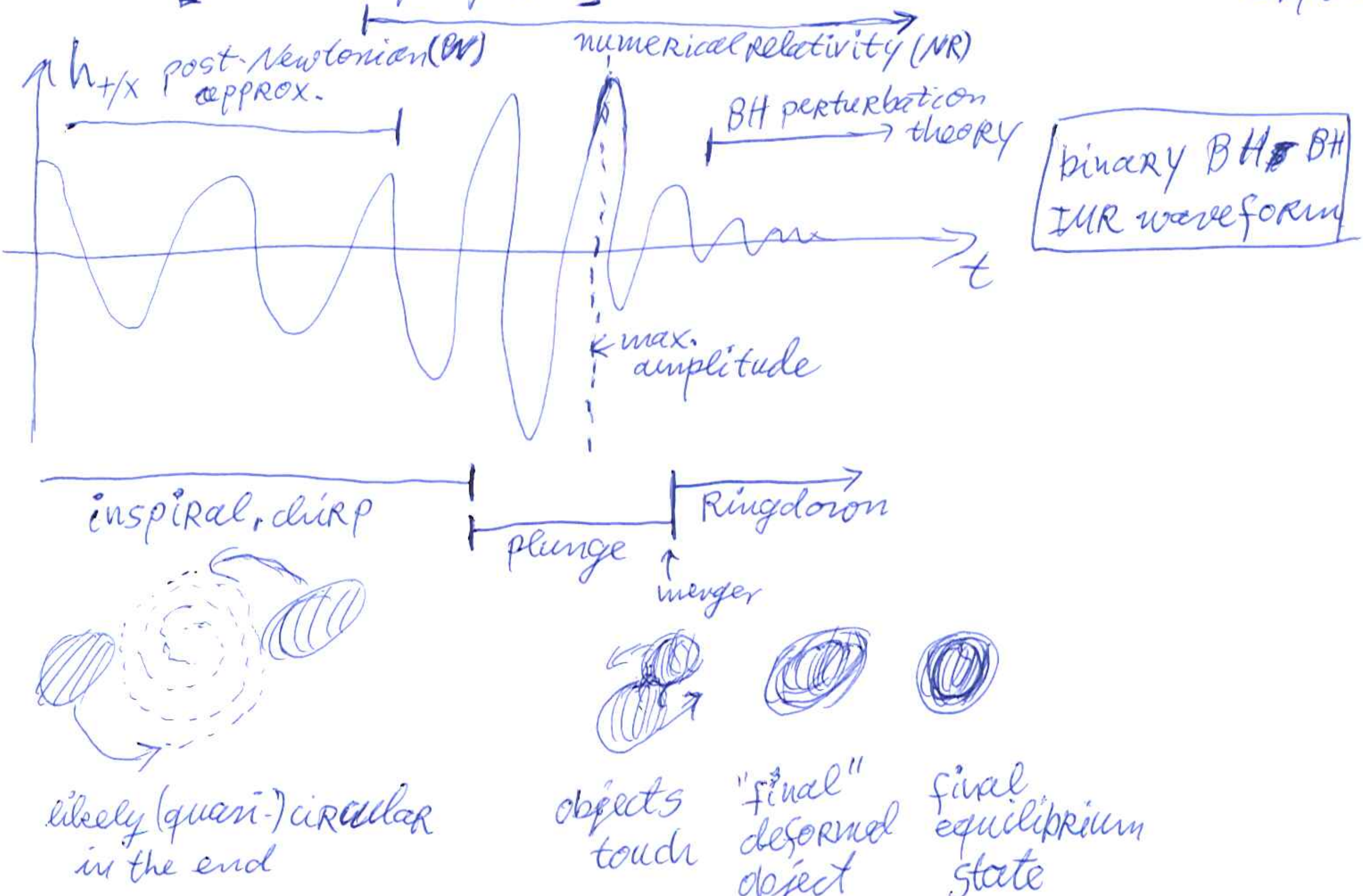
Here: gravitational waves (GW) [jan-steinhoff.de/lectures/ictp2018/]

from compact binary coalesence, CBC [in GR]

compact binary → black holes (BHs), neutron stars (NSs) [exotic comp. obj. ECO]

coalesence → inspiral, merger, and Ringdown (IMR)

[also: cont. waves, supernovae, bigbang, stochast. backgr., surprises?]



binary NS-BH: - tidal disruption of NS → no Ringdown
- no 〃 → similar to BH-BH

binary NS-NS: complicated post-merger waveform, but with characteristic frequencies

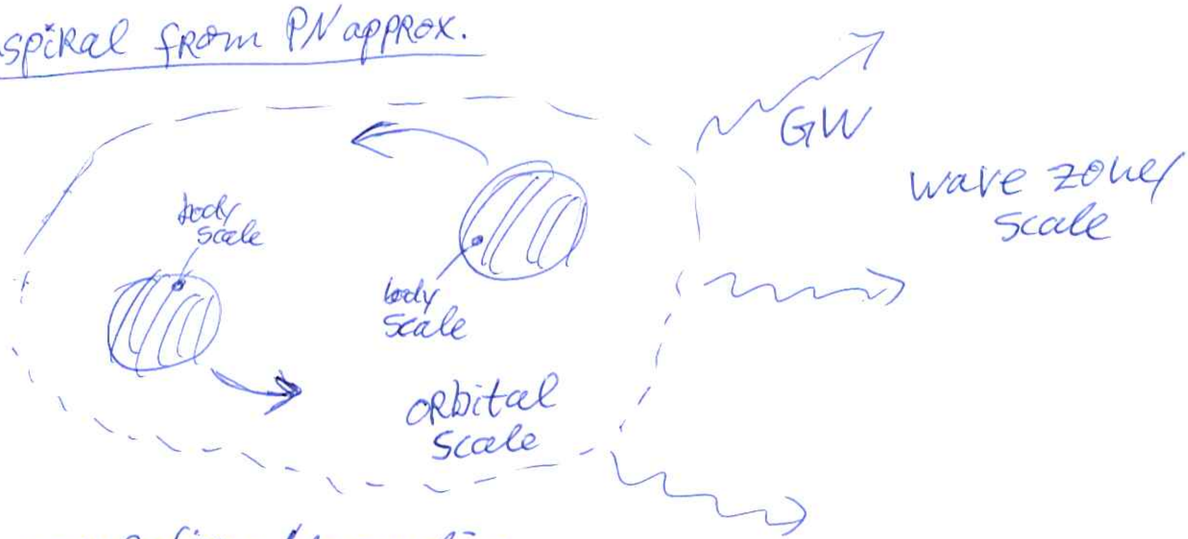
IMR waveform models → combine PN & NR

phenomenological: emphasis on low comp. cost
EOB: emphasis on physics
} not always a clear cut

[Reduced order models: waveform compression & interpolation]

ICTP18 p.2.

inspiral from PN approx.



$\rightsquigarrow$ separation/decoupling
of scales $\rightsquigarrow$ approximation possible

still complicated: - spin, multipoles
- tidal: deformation
heating
resonances
- tail effect

scale $\downarrow$

~~assume:~~ assume: circular orbit ($r \approx$ const)

PN approximation:

$$x := (M\omega)^{2/3} \sim \frac{M}{r} \sim v \underline{\underline{\ll 1}}$$

3rd Kepler
$\omega^2 r^3 \sim M$

$\boxed{G=1=c}$

$M = m_1 + m_2$ total mass

$\mu = \frac{m_1 m_2}{M}$ reduced mass

$\nu = \frac{\mu}{M}$, $0 \leq \nu \leq \frac{1}{4}$ symmetric mass ratio

test-mass, equal masses

PN phase of GW ϕ:

binding energy ~ (energy − M)/μ

$\hookrightarrow e(x) = -\frac{1}{2}x\left[1 - \frac{1}{12}(9+\nu)x + \dots\right]$

known to x^4 (4PN) [gauge indep!]

GW luminosity

$\hookrightarrow \mathcal{L} = \frac{32}{5}\nu^2 x^5\left[1 - \left(\frac{1247}{336} + \frac{35}{12}\nu\right)x + \dots\right]$

known to $x^{7/2}$ (3.5PN)

energy balance: [adiabatic approx.]

$$\mu \frac{de}{dt} = -\mathcal{L} \quad \Rightarrow \quad \frac{dx}{dt} = -\frac{\mathcal{L}}{de/dx}\cdot\frac{1}{\mu}$$

$$\frac{d\phi}{dt} = \omega \quad \rightsquigarrow \quad \frac{d\phi}{dt} = \frac{x^{3/2}}{M}$$

$\rightsquigarrow$ analytic & numeric solutions
$\rightsquigarrow$ TaylorT? models

Taylor F2: - analytic sol. $\phi(t)$ in $h_{+/\times}$
- analalytic Fourier trafo (stationary phase approx.)

shortcomings: - motion not exactly spherical
$\rightsquigarrow$ plunge!
- asymptotic series $\rightsquigarrow$ need resummation!
e.g. use test-body case: $e(x) = \frac{1-2x}{\sqrt{1-3x}} - 1 \approx -\frac{1}{2}x\left(1 - \frac{3}{4}x + \dots\right)$

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PN amplitude of GW

R: distance to source

$$h_+ = -\frac{4\mu}{R} \cdot \frac{1+\cos^2\theta}{2} \cdot \cos(2\phi) \cdot x + \dots$$

$$\boxed{\omega_{GW} = 2\omega \text{ !}}$$

$$\underbrace{h_\times = -\frac{4\mu}{R} \cdot \cos\theta \cdot \sin(2\phi) \cdot x}_{\text{"restricted" waveform}} + \underbrace{\dots}_{\text{PN correction}}$$

PN corrections only in $\phi(t), x(t)$

a point about ~~poaint~~ point masses: ($\approx$ nonrotating BHs)

~~action of a binar~~

action $S_{\text{pointmass}} = m\int \underset{\substack{\uparrow \\ \text{proper time}}}{d\tau} = \int \underset{\substack{\uparrow \\ \text{affine parameter}}}{d\sigma}\, \underbrace{m\sqrt{g_{\mu\nu} u^\mu u^\nu}}_{-L}$, $\quad u^\mu = \frac{d x^\mu(\sigma)}{d\sigma}$ (worldline)

EOM $\delta S = 0 \Rightarrow \frac{D}{d\sigma}\Big(\underbrace{\frac{m u_\mu}{\sqrt{-u_\nu u^\nu}}}_{p_\mu = \frac{\partial L}{\partial u^\mu}}\Big) = 0$ geodesic equation

~~the~~ canonical 4-momentum

now: Legendre trafo

but: $L = p_\mu u^\mu \Rightarrow H = p_\mu u^\mu - L = 0$?

~~need to add~~ map $u^\mu \to p_\mu$ not invertible

need to add constraint $p_\mu p^\mu + m^2 = 0$ to action

[from Nambu-Goto to Polyakov]

$\Rightarrow S_{\text{pointmass}} = \int d\sigma \left[p_\mu u^\mu - \lambda (g^{\mu\nu} p_\mu p_\nu + m^2) \right]$

$\lambda(\sigma)$: Lagrange multiplier, "worldline metric"

EOM: $p^2 = -m^2$, $u_\mu = 2\lambda p_\mu$, $\boxed{\frac{Dp_\mu}{d\sigma} = 0}$

$\hookrightarrow \lambda = \frac{1}{2m}\sqrt{-u_\nu u^\nu} \Rightarrow p_\mu = \frac{m u_\mu}{\sqrt{u_\nu u^\nu}}$

mass-shell $g^{\mu\nu} p_\mu p_\nu = -m^2$ encodes dynamics !

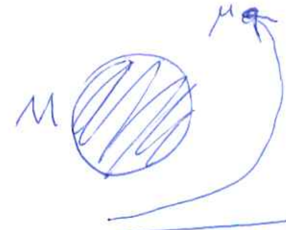
takes place of Hamiltonian !

"interaction" linear in $g^{\mu\nu}$!

example: test-mass in Schwarzschild metric
↳ mass μ ↳ mass M

$$(x^\mu) = \begin{pmatrix} t \\ r \\ \theta \\ \phi \end{pmatrix}$$

assume: $\boxed{\theta = \frac{\pi}{2}}$



gauge choice: $\sigma = t$ = coord. time

$$\Rightarrow u^0 = \frac{dt}{d\sigma} = 1 \Rightarrow p_\mu \dot{x}^\mu = p_r \dot{r} + p_\phi \dot{\phi} + p_0$$

solve mass-shell for $H := -p_0$

$$g^{\mu\nu} p_\mu p_\nu = -\mu^2$$

with $d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu$

$$= A\,dt^2 - \frac{dr^2}{A} - r^2(\overset{=0}{d\theta^2} + \overset{=1}{\sin^2\theta}\, d\phi^2),$$

$$\boxed{A = 1 - \frac{2M}{r}}$$

$$\Rightarrow -\frac{1}{A} H^2 + A p_r^2 + \frac{L^2}{r^2} = -\mu^2, \quad p_\phi \equiv L = \text{angular momentum}$$

$$\boxed{H = \sqrt{A \cdot \left(\mu^2 + A p_r^2 + \frac{L^2}{r^2}\right)}}$$

exercise: show that $\omega^2 r^3 = M$ for circular orbits

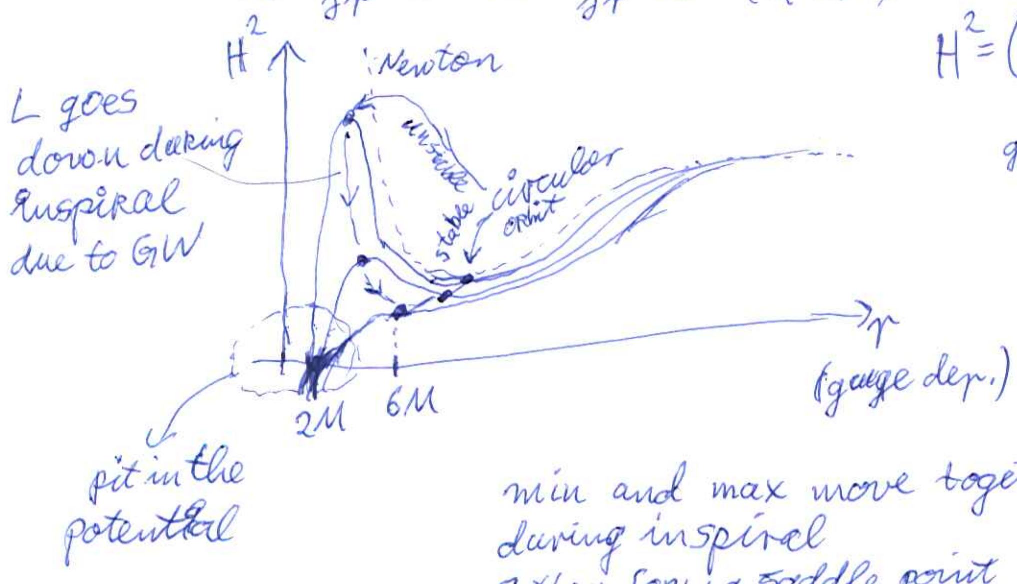
action: $S_{\text{point mass}} = \int dt \left(p_r \dot{r} + L \cdot \dot{\phi} - H\right)$

EOM: $\frac{dr}{dt} = \frac{\partial H}{\partial p_r}$, $\frac{dp_r}{dt} = -\frac{\partial H}{\partial r}$

$\omega \equiv \frac{d\phi}{dt} = \frac{\partial H}{\partial L}$, $\frac{dL}{dt} = -\frac{\partial H}{\partial \phi} = 0 \Rightarrow L = \text{const}$ [changes with radiation]

now: circular orbits, $p_r = 0$, $r = \text{const}$

$$\Rightarrow \frac{\partial H}{\partial r} = 0 \quad \text{or} \quad \frac{\partial H^2}{\partial r} = 0 \quad (\text{if } H \neq 0)$$



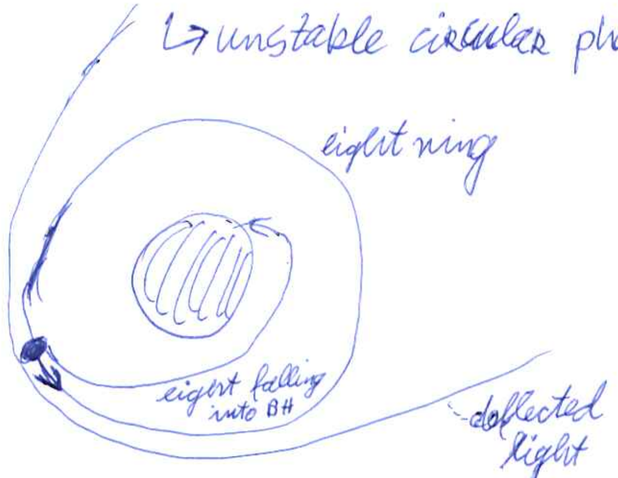
$$H^2 = \underbrace{\left(1 - \frac{2M}{r}\right)}_{\text{grav. attraction}} \underbrace{\left(\mu^2 + \frac{L^2}{r^2}\right)}_{\text{angular mom. "barrier"}}$$

min and max move together during inspiral
⇒ they form a saddle point
⇒ no stable orbit any more for $r \leq 6M$ — innermost stable circular orbit (ISCO)
⇒ mass μ plunges into BH

gravity wins!

another strong-field feature: light ring

↳ unstable circular photon orbit



$$H^2(\mu\to 0) = A\cdot\frac{L^2}{r^2}$$

$$0 = \frac{\partial H^2_{\mu=0}}{\partial r} \Leftrightarrow 0 = A'\frac{L^2}{r^2} - \frac{2L^2}{r^3}A \qquad {}' = \frac{d}{dr}$$

$$\boxed{2A = rA'} \Leftrightarrow 2\left(1-\frac{2M}{r}\right) = \frac{2M}{r}$$

$$\Leftrightarrow \boxed{r = 3M}$$

also: point of max. ~~frequency~~ amplitude for orbiting mass μ

$$\cancel{0 = \frac{\partial \omega}{\partial r} = \frac{\partial H}{\partial r \partial L}} \text{ or } \cancel{0 = \frac{\partial H^2}{\partial r \partial L}}$$

↳ ≈ half of the GW fall into the BH

$$\cancel{0 = \frac{d\omega}{dt}}$$

⇝ attach ringdown ≈ at light ring

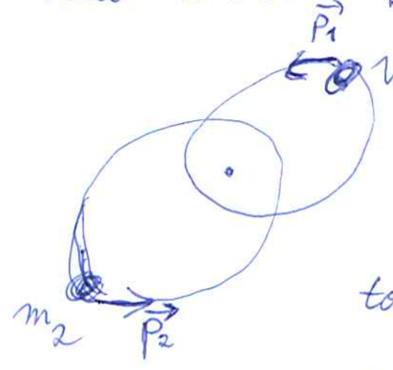
Effective-one-body (EOB) model [Buonanno & Damour 1998]

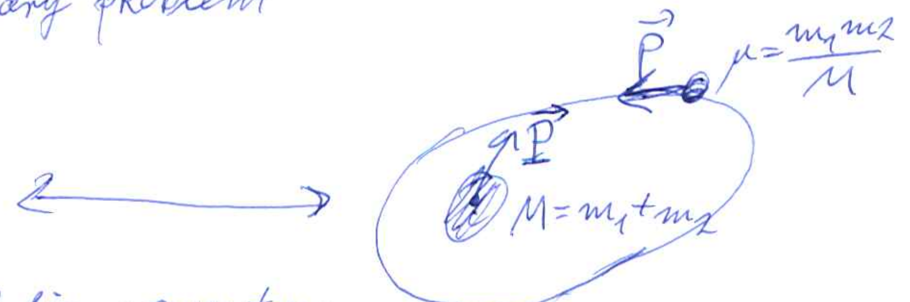
- combine PN results with test-mass motion → strong-field effects (ISCO, light ring)
- attach ringdown (BH perturbation theory, also NR) ≈ at light ring

  recent approach: backwards one body (BOB for waveform from ≈ ISCO to end
- calibrate to match numerical relativity

combine PN & test-mass

recall Newtonian binary problem





total lin. momentum $\vec{P} = \vec{P}_1 + \vec{P}_2 = \text{const}$

Relative momentum $\vec{p} = \vec{P}_1 = -\vec{P}_2$ in center-of-mass (COM) system, $\vec{P} = 0$

Relativistic: mass-shell $g^{\mu\nu}p_\mu p_\nu = -m^2$ encodes dynamics!

idea: map $p_{1/2}^2 = -m_{1/2}^2$ to mass-shells for $P^\mu = p_1^\mu + p_2^\mu$, $p^\mu = ?$

assume $g_{\mu\nu} = \eta_{\mu\nu}$ for now

define $p_\mu := \frac{1}{M}\begin{pmatrix} +p_{1\mu} p_2^\mu \\ P_0 \vec{p}_1 \end{pmatrix}$ in COM frame, $\vec{p}_1 = -\vec{p}_2$

then: ~~(exercise)~~

$p_\mu p^\mu = \mu^2$ $\qquad \nu = \mu/M$

$\rightarrow P_\mu P^\mu = M_P^2 := M^2\left[1 + 2\nu\left(-\frac{p_0}{\mu} - 1\right)\right]$ energy map $P_0 \leftrightarrow p_0$ (exercise)

$H = P_0 =$ "real" Hamiltonian/energy

$H_e = -p_0 =$ "effective" Hamiltonian/energy

$\Rightarrow H = \sqrt{M^2\left[1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)\right] + \vec{P}^2}$ from kinematics, no interaction!

$\vec{P} = 0$: effective-one-body approximation (neglect recoil)

H_e for grav. binary?

- calculate PN result H_{PN}
- match $H \hat{=} H_{PN}$ mod. canonical trafo

$\Rightarrow H_e$ up to "coordinate" freedom

Methods for matching:

- compare gauge-invariant quantity, eg.,
  - circular-orbit binding energy $e(x)$
  - Delaunay Hamiltonian $\hat{=}$ relativistic gravitational "Balmer" formula
  - scattering angle/amplitude
- explicitly construct a canonical transformation

Idea: use deformed test-body Hamiltonian for H_e suggested by Newtonian case

$\hookrightarrow$ deformed metric and mass-shell for p_μ (effective)

$$g^{\text{eff}}_{\mu\nu} dx^\mu dx^\nu = -A dt^2 + \frac{dr^2}{A \cdot \bar{D}} + \frac{r^2}{C}(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$g^{\mu\nu}_{\text{eff}} p_\mu p_\nu + Q = -\mu^2$$

here $A, \bar{D}, C$ are fct. of r, ν, and Q is fct of r, ν, p_i

gauge/coordinate choice: $C=1$, $Q = Q(r, p_r, \nu)$

~~convenient: $Q \approx 0$ for quasi-circular ins~~

$\rightsquigarrow$ $H_e = \sqrt{A \cdot (\mu^2 + A\bar{D} p_r^2 + \frac{L^2}{r^2}) + Q}$, $H = \sqrt{M^2\left[1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)\right]}$, $\vec{P} = 0$

(energy map)

EOB potentials: $u := \frac{M}{r}$

$$A = 1 - 2u + 0\,u^2 + 2\nu \cdot u^3 + \left(\frac{94}{3} - \frac{41\pi^2}{32}\right)\nu \cdot u^4 + (\ldots\nu + \ldots \nu \ln u + \ldots \nu^2) \cdot u^5$$

$$\bar{D} = 1 + 0 \cdot u + 6\nu \cdot u^2 + (52\nu - 6\nu^2) \cdot u^3 + (\quad " \quad) \cdot u^4$$

$$Q = \underbrace{}_{0PN} \underbrace{}_{1PN} \underbrace{}_{2PN} \underbrace{2(4-3\nu)\nu \cdot u^2 p_r^4}_{3PN} + \underbrace{\ldots \text{complicated} \ldots}_{4PN}$$

- for circular orbits $p_r = 0 \rightsquigarrow Q = 0$, $\bar{D}$ irrelevant
  $\rightsquigarrow A(r, \nu)$ determines dynamics ∎

- g_{eff} is Schwarzschild metric at 1PN ∎
- ν-deformed metric at 2PN
- at 3PN, 4PN: deformed g_{eff} and mass-shell (nongeodesic motion in g_{eff}) ← [Finsler geometry] resummations: - Padé - log; calibration parameters ∎

exercise: check the circular binding energy $e(x)$ to 1PN from H. with Mathematica

Radiation reaction

so far: conservative dynamics
↳ add radiation-reaction force to EOM
force ~ $\dot{p}$ EOM

$$\dot{r} = \frac{\partial H}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H}{\partial r} + f_r$$

$$\omega = \dot{\phi} = \frac{\partial H}{\partial L}, \quad \dot{L} = -\underbrace{\frac{\partial H}{\partial \phi}}_{0} + f_\phi \quad \text{angular mom. loss ∎}$$

$\rightsquigarrow$ energy loss: $\dot{E} = \frac{dH}{dt} = \dot{r}\frac{\partial H}{\partial r} + \dot{\phi}\underbrace{\frac{\partial H}{\partial \phi}}_{0} + \underbrace{\dot{p}_r \frac{\partial H}{\partial p_r}}_{(-\frac{\partial H}{\partial r} + f_r)\dot{r}} + \underbrace{\dot{L}}_{f_\phi}\underbrace{\frac{\partial H}{\partial L}}_{\dot{\phi}}$

$$= \dot{r} f_r + \dot{\phi} \cdot f_\phi$$

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energy & angular momentum balance
↳ only hold ~~on~~ on average, not instantaneous!
↳ Schott-terms

$$\dot{E} + \dot{E}_{\text{Schott}} + \mathcal{F}_E = 0 \qquad \mathcal{F}_E \equiv \mathcal{L}$$

$$\dot{L} + \dot{L}_{\text{Schott}} + \underbrace{\mathcal{F}_L}_{\text{fluxes at infinity}} = 0 \qquad \langle \dot{E}_{\text{Schott}} \rangle = 0 = \langle \dot{L}_{\text{Schott}} \rangle$$

$$\rightsquigarrow \dot{r} f_r + \dot{\phi} \cdot f_\phi + \dot{E}_{\text{Schott}} + \mathcal{F}_E = 0$$

$$f_\phi + \dot{L}_{\text{Schott}} + \mathcal{F}_L = 0$$

Solution for f_r, f_ϕ: see Bini & Damour (2012) ← see also

here: assume quasi-circular inspiral

$\rightsquigarrow$ ~~neglect~~ drop f_r * average irrelevant ($\dot{L}_{\text{Schott}} \equiv 0 \equiv \dot{E}_{\text{Schott}}$)

$$\rightsquigarrow f_\phi = \mathcal{F}_L = \frac{\mathcal{F}_E}{\dot{\phi}} = \frac{\mathcal{L}(x)}{\omega}, \quad f_r = 0$$

*not independent from f_ϕ, but cannot have $f_\phi = 0$ $\rightsquigarrow$ choice $f_r = 0$

can be obtained from PN, but should be resummed!

but we also need the GW modes:

$$h_+ + i h_\times = \sum_{\ell, m} h_{\ell m} \underbrace{{}_{-2}Y^{\ell m}}_{\text{spin-weighted spherical harmonics}}$$

$$\text{then: } \mathcal{L} \sim \oint d\Omega \, (\dot{h}_+^2 + \dot{h}_\times^2) \sim \sum_{\ell m} |\dot{h}_{\ell m}|^2$$

$\boxed{\rightsquigarrow \text{resum } h_{\ell m}}$ $\rightsquigarrow$ get $\mathcal{L}$ $\rightsquigarrow$ $\boxed{\text{get } f_\phi}$ $\qquad x = (M\omega)^{2/3}$

Resummation through factorization:

$$h_{\ell m} = \underbrace{h^N_{\ell m}}_{\text{"Newtonian" (leading-order)}} \cdot \underbrace{\hat{h}_{\ell m}}_{\text{PN correction } 1 + \ldots} = h_{\ell m}(x) \quad \text{circular orbit}$$

$\hat{h}_{\ell m}$ is again factorised: $\hat{h}_{\ell m} = S^{(\epsilon)}_{\text{eff}} \, T_{\ell m} \, e^{i\delta_{\ell m}} \underbrace{(\rho_{\ell m})^\ell}_{\text{inspired by } h_{\ell m} \sim x^\ell_{\text{harmonic}}(x)}$

justification: WORKS well

effective source $S^{\ell m}_{\text{eff}}$
$\sim$ GW propagate in field of binary

$$S^{\ell m}_{\text{eff}} = \begin{cases} H_{\text{e}}/\mu & \ell + m \text{ even} \\ L/\sqrt{x}\,\mu M & \ell + m \text{ odd} \end{cases}$$

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Resummed "leading logarithms"

$$T_{\ell m} = \frac{\Gamma(\ell+1-2ik)}{\Gamma(\ell+1)} e^{\pi k} e^{2ik \log(2kr_0)} \; ; \quad r_0 = \frac{2M}{\sqrt{e}}$$

$k = H \cdot m \cdot \omega$ (H = ADM mass)

the "rest": $e^{i\delta_{\ell m}}$

subleading term, including logs

but: ~~not~~ circular approximation $p_r \approx 0$ not good during late inspiral

$\leadsto$ non quasi-circular (NQC) correction $f_{\ell m}^{NQC}$ + tweaks here and there...

$$h_{\ell m} \to f_{\ell m}^{NQC} h_{\ell m}$$

ansatz with parameters matched to NR

recall: EOB potentials also have matching param.

matching to NR $\approx$ parameter estimation (data analysis) [figure of merit?]

status: waveform model for inspiral + plunge

missing: ringdown $\triangleq$ perturbed BH (final, merged)

BH perturbations (linear order)

$$g_{\mu\nu} = g_{\mu\nu}^{back} + g_{\mu\nu}^{pert}$$

g^{back} here: Schwarzschild; g^{pert} small

spherical* symmetry of g^{back} *and time ~~translat.~~ independence

$\leadsto g_{\mu\nu}^{pert} \sim$ (radial fct) $\times$ (vector spherical harm.) $\times e^{-i\omega t}$

(linear) ~~master eq~~ scalar master eq. for rad. parts (in Regge-Wheeler gauge for g^{pert})

- Zerilli eq. (even)
- Regge-Wheeler eq. (odd)

variable trafo (Chandrasekhar) "electric-magnetic" duality

$\leadsto$ only need Regge-Wheeler (RW) eq.:

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_{RW}\right] Q(r_*) = 0$$

Q: RW master fct

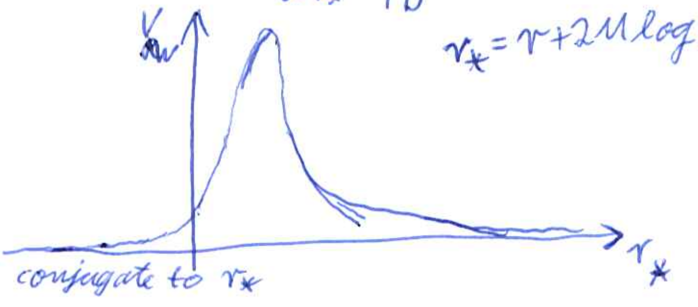
$$V_{RW} = A\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right), \quad \text{"tortoise" coord.} \quad \frac{dr}{dr_*} = \frac{A}{\sqrt{\bar{D}}}$$

here: $\bar{D} = 1$, $A = 1 - \frac{2M}{r}$

$r_* = r + 2M \log(r - 2M$

$r \in [2M, \infty]$

$\hookrightarrow r_* \in [-\infty, \infty]$

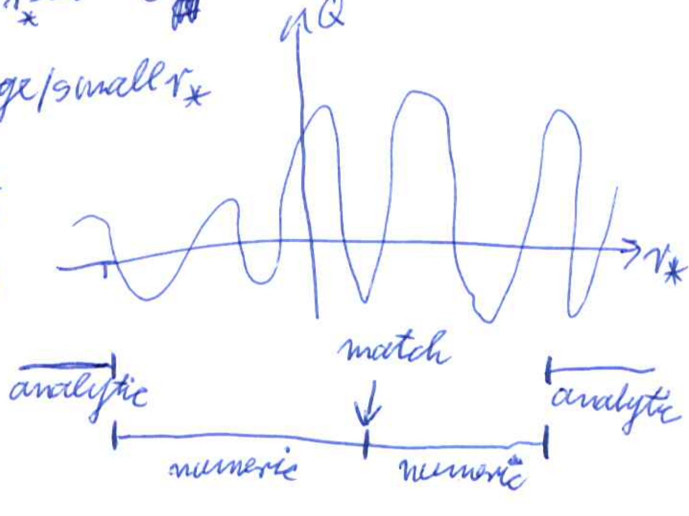


Remark: EOB pot. and EOM actually written in terms of p_{r_*} conjugate to r_*

ICTP18 p.10 (skipped)

solutions:

- analytic: infinite series of hypergeom. fcts.
- numeric: ordinary diff. eq.
  eg. "direct integration"
  1.) analytic solution around $r_* = \pm\infty$: $Q \sim (\text{series}) \cdot e^{\pm i\omega r_*}$
  2 integ. const. $C_\pm$ for each case $r_* = \pm\infty$
  2.) numeric solution from large/small r_* to some $r_* = r_*^{match}$
  3.) fix 2 of 4 integr. const. by matching Q, Q' at r_*^{match}



↳ mixed numeric/analytic solution with 2 integ. const.

physical/black hole boundary condition: no wave coming from horizon at $r_* = -\infty$ ⇒ $Q \sim e^{-i\omega r_*}$

⇒ fixes one int. const., one left

⇒ remaining int. const. is overall amplitude (physically irrelevant for linear perturbation)

quasi-normal modes (QNM)

FREE oscillations of BHs

⇒ no incoming wave from $r_* = \infty$ ⇒ $Q \sim e^{+i\omega r_*}$

⇒ only possible for complex ω (discrete ω^{QNM} FREQUENCIES)

⇒ damped sinusoid $\sim e^{-i\omega t}$, $\omega \in \mathbb{C}$

→ derivations and codes: jan-steinhoff.de/lectures/ictp2018/

final/merged BH is rotating ⇒ $g_{\mu\nu}^{back}$ is Kerr metric

⇒ Teukolski master eq. for radial part, spheroidal harmonics

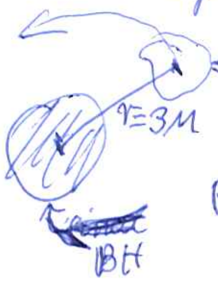
⇒ solution methods essentially the same

The fundamental QNM

↳ dominates the ringdown, ×of final BH
least-damped mode, ×smallest $\mathrm{Im}(\omega)$ → numerical sol from RW-eq: $M\omega_{RD} = 0.38\ldots + i\ldots$ ↑ ringdown

estimate from light-ring physics:
plunging bodies create curvature perturbations, which can linger close to the light-ring orbit of the final BH (unstable)

light-ring orbit ≙ resonator



← curvature perturbation

$$\mathrm{Re}(\omega_{RD}) \approx 2\omega \approx \frac{2}{M}\left(\frac{M}{r}\right)^{3/2} \approx \frac{2}{3\sqrt{3}M}$$

$r = 3M$

↑ orbital freq. of perturber $\omega_{GW} \approx 2\omega$

"3rd Kepler" $\omega^2 r^3 = M$, $(M\omega)^2 = \left(\frac{M}{r}\right)^3$

↝ $\mathrm{Re}(M\omega_{RD}) \approx 0.38\ldots$ fits quite well!

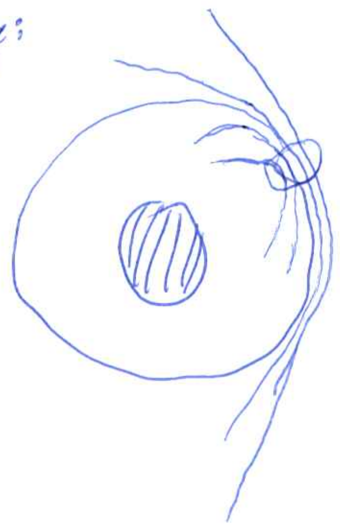
can ~~be~~ extend this estimate to linear order in the BH's spin (use the Kerr metric).

Lyapunov exponent of geodesic congruence around the light-ring (unstable)
⟷ damping time of ringdown, $\mathrm{Im}(\omega_{RD})$

Backwards-One-Body merger-Ringdown model
arXiv:1810.00040 McWilliams

extrapolate the picture above backwards in time
congruence of light rays close to light ring:

$$r = r_{LR}\left[1 + \varepsilon \cdot \sinh(\gamma (t - t_p)) + O(\varepsilon^2)\right]$$
$$\phi = \omega t + O(\varepsilon^2)$$
$$\theta = \tfrac{\pi}{2} + O(\varepsilon^2)$$



γ: Lyapunov exponent
t_p: congruence converges
~ perturber is created ~ mass, GW amplitude

only expansion in r-direction
by $dr \propto \cosh[\gamma(t-t_p)]$
↝ perturber gets diluted as $\cosh^{-1}[\gamma(t-t_p)]$
↝ GW "amplitude" evolves as $|\Psi_4| = A_p \cosh^{-1}[\gamma(t-t_p)]$
↑ Weyl scalar "$\ddot{h}$"

FROM amplitude to phase?

↳ implicit Rotating source (IRS) picture

L: angular momentum of pert.

final BH $\sim e^{-\gamma t}$ + const

ω_{GW}: frequency of emitted GW $\sim e^{-\gamma t}$ + const

⇝ $I := \underbrace{\frac{dL}{d\omega}}_{\text{"moment of inertia"}} \approx$ const   confirmed by numerical simulations

energy balance $|\dot{h}|^2 \propto \frac{d}{dt}(I\omega^2) = 2I\omega\dot{\omega}$ ← secular time derivative

also: $\dot{h} \sim \frac{1}{\omega}\ddot{h} \sim \frac{1}{\omega}\psi_4$ ← here: $\frac{d}{dt} \sim \omega$

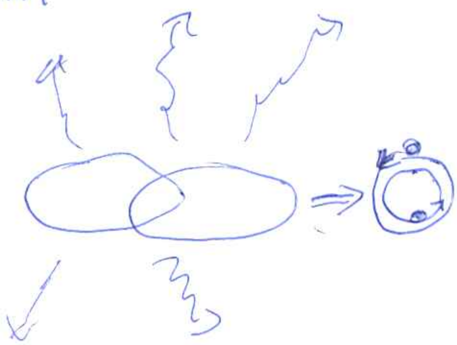
⇝ $\omega \cdot \dot{\omega} \propto \left|\frac{\psi_4}{\omega}\right|^2 \propto \frac{1}{\omega^2}\cosh^{-2}[\gamma(t-t_p)]$

⇝ integrate ⇝ $\omega(t) \propto [\tanh(\gamma(t-t_p)) + \text{const}]^{1/4}$

integrate $\omega = \dot{\phi}$ ⇝ phase $\phi(t)$ (complicated, but analytic)

proportionality/integration constants fixed by matching to inspiral waveform

Review: Physics of IMR waveforms



- early inspiral: orbit circularization ("quasi-equilibrium") due to GW emission
- late inspiral ≈ circular orbit
  amplitude + frequency grow: "chirp" until ISCO at $r \sim 6M$

- ISCO: gravity wins ⇝ plunge (circular motion becomes unstable)
- light-ring, $r \sim 3M$
  GW start to fall into the other/final BH
  ≈ half of the GW fall in
  ≈ point of max. amplitude

- common event horizon forms, $r \sim 2M$
  $\hookrightarrow$ merger!
- Ringdown of final deformed BH
  GW "resonate" within light-ring
  of final BH
- final (rapidly) spinning BH, recoil motion