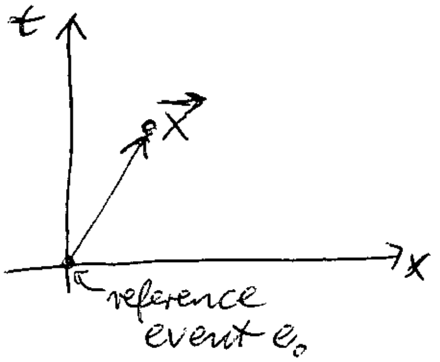
**Def.:** a vector is an element of a $\underbrace{\text{vector space}}$
↳ defined as usual

also as usual: decomposition in basis vectors $\vec{e}_\mu$, $\mu = 0, 1, 2, 3$
↑ time

example: coordinate vector




$$\vec{X} = \sum_{\mu=0}^{3} X^\mu \vec{e}_\mu = X^\mu \vec{e}_\mu$$

↑ Einstein's summation convention

↑ components in basis $\vec{e}_\mu$: $X^0 = t$, $X^1 = x$, $X^2 = y$, $X^3 = z$

but: $\vec{X}$ is independent of the choice of basis! [no basis in def. of vector]

**Def:** linear form or 1-form $f$ is a linear map from vector to number

$$f: \text{vector} \longrightarrow \text{number}, \quad f(a\vec{x} + b\vec{y}) = a f(\vec{x}) + b f(\vec{y})$$

space of linear forms: (dual) vector space
↳ choose a basis $\omega^\mu$: require $\omega^\mu(\vec{e}_\nu) = \delta^\mu_\nu$ ⇔ $\omega^\mu$ dual basis of $\vec{e}_\mu$
↳ basis decomposition: $f = f_\mu \omega^\mu$
↑ components, ↑ basis

then: $\underline{f(\vec{X})} = f_\mu \omega^\mu(\vec{X}) = f_\mu \omega^\mu(X^\nu \vec{e}_\nu)$

$$= f_\mu X^\nu \underbrace{\omega^\mu(\vec{e}_\nu)}_{\delta^\mu_\nu} = \underline{\underline{f_\mu X^\mu}}$$

makes sense [form maps vector and vice versa, generalization?]

**Def:** a $(p,q)$-tensor is a linear map from $p$ linear forms and $q$ vectors to a number: $T(f_1, \ldots, f_p, \vec{x}_1, \ldots, \vec{x}_q) = \text{number}$
↳ geometric object, no basis or components needed to define it!
↳ basis decomposition:

$$T(f_1, \ldots, f_p, \vec{x}_1, \ldots \vec{x}_q) = T(f_{1\mu_1}\omega^{\mu_1}, \ldots, f_{p\mu_p}\omega^{\mu_p}, x_1^{\nu_1}\vec{e}_{\nu_1}, \ldots, x_q^{\nu_q}\vec{e}_{\nu_q})$$

$$= f_{1\mu_1} \cdots f_{p\mu_p} x_1^{\nu_1} \cdots x_q^{\nu_q} \underbrace{T(\omega^{\mu_1}, \ldots, \omega^{\mu_p}, \vec{e}_{\nu_1}, \ldots, \vec{e}_{\nu_q})}_{T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}}$$

$T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}$: components of $T$ in basis $\vec{e}_\nu$, $\vec{\omega}^\mu$

more definitions:

- upper indices: ~~Ko~~ntravariant indices
  lower " : covariant "
- tensor field: map of spacetime point to tensor
- $n$-form: completely antisymmetric $(0,n)$-tensor
- differential form: $n$-form field

example: scalar product $\vec{x}\cdot\vec{y} \equiv g(\vec{x},\vec{y})$ = number, is linear

$\rightsquigarrow g$ is a $(0,2)$-tensor $\rightsquigarrow$ metric tensor

symmetric $g(\vec{x},\vec{y}) = g(\vec{y},\vec{x}) \rightsquigarrow g_{\mu\nu} = g_{\nu\mu}$

inverse metric $g^{\mu\nu}$: inverse matrix of $g_{\mu\nu}$

$\hookrightarrow g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu}$

"pulling indices" convention: $x_\mu \equiv g_{\mu\nu} x^\nu$, $x^\nu \equiv g^{\nu\mu} x_\mu$

bijection vectors $\longleftrightarrow$ 1-forms

analogous for tensors

orthonormal Lorentz basis:

one can choose a basis such that

$$g(\vec{e}_\mu,\vec{e}_\nu) = g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \equiv \eta_{\mu\nu}$$ Minkowsky metric

(diagonalization of ~~g~~ symmetric matrix & rescaling)

$$\rightsquigarrow \vec{x}\cdot\vec{y} = \eta_{\mu\nu} x^\mu y^\nu = -x^0 y^0 + \underbrace{x^1 y^1 + x^2 y^2 + x^3 y^3}_{\text{Euclidean scalar product}}$$

writing $(x^\mu) = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ : $\vec{x}\cdot\vec{x} = -t^2 + x^2 + y^2 + z^2$ = invariant $\nabla$ (lecture 1) under change of inertial system with same origin

## Lorentz trafo:

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change of inertial system must leave scalar product/metric invariant $\rightsquigarrow$ isometry

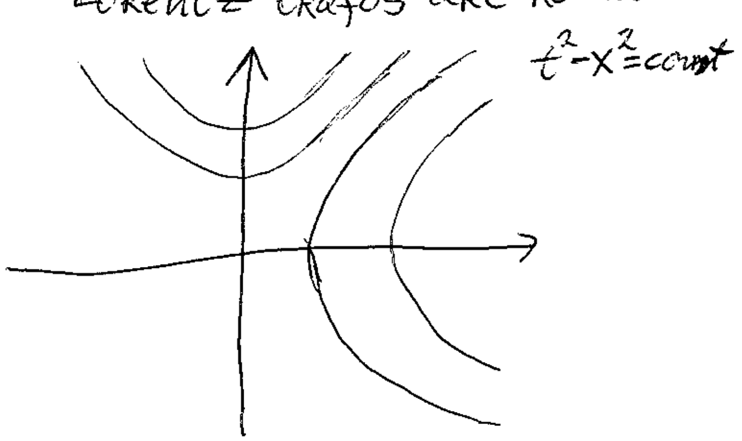
$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (*) \quad \text{such that} \quad \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu \quad \forall \vec{x}$$

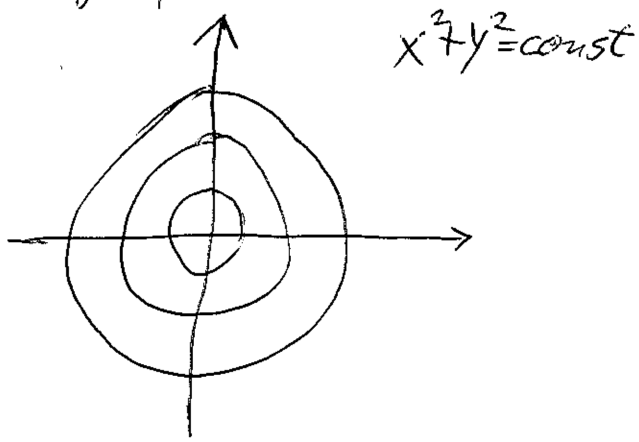
$\Lambda$ ↑ Lorentz matrix

$$\rightsquigarrow \boxed{\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}}$$

compare to Rotations $\delta_{ij} R^i{}_k R^j{}_l = \delta_{kl}$ or $R^T R = \mathbb{1}$

Lorentz trafos are "Rotations" along hyperbola




$t' = \cosh\alpha \cdot t - \sinh\alpha \cdot x$

$x' = -\sinh\alpha \cdot t + \cosh\alpha \cdot x$

$\rightsquigarrow t'^2 - x'^2 = t^2 - x^2$

$v = \tanh\alpha = \frac{-x'}{t'}$ for $x = 0$

$\rightsquigarrow \cosh\alpha = \gamma, \ \sinh\alpha = v \cdot \gamma$

$\rightsquigarrow t' = \gamma t - \gamma v x \qquad y' = y$

$x' = -\gamma v t + \gamma x \qquad z' = z$ } compare to (*)

$x' = \cos\theta \cdot x + \sin\theta \cdot y$

$y' = -\sin\theta \cdot y + \cos\theta \cdot x$

$\rightsquigarrow x'^2 + y'^2 = \cancel{\text{const}}\ x^2 + y^2$

$$(\Lambda^\mu{}_\nu) = \begin{array}{c} \\ \mu \downarrow \end{array} \overset{\xrightarrow{\nu}}{\begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & 0 & \\ & 0 & 1 & \\ & & & 1 \end{pmatrix}}$$

Lorentz matrix for boost in x-direction

## Why tensors?

inertial frames $\hat{=}$ orthonormal basis

principle of special relativity

↳ laws of electrodyn./laws of physics in inertial frames should be written as tensor equations on spacetime

$\rightsquigarrow$ geometry of tensors on spacetime is manifestly independent of a specific basis/inertial frame!