Why tensors? Tensor analysis in SR (continued) IMPRS 2018 L3P1

↳ Principle of special relativity

↳ geometry of tensors on spacetime is manifestly independent of a specific basis, i.e. inertial frame

→ need to find spacetime (tensor) versions of physical quantities

Velocity (4-velocity)

$$u^\mu = \frac{dx^\mu}{d\tau} \quad \tau: \text{proper time} \qquad (x^\mu) = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$= \underbrace{\frac{dt}{d\tau}}_{\gamma} \frac{dx^\mu}{dt} \quad \leadsto \quad (u^\mu) = \gamma \begin{pmatrix} 1 \\ v^x \\ v^y \\ v^z \end{pmatrix} \Big\} \text{ordinary velocity } v$$

$$u^\mu u_\mu = \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{d\tau^2} = \frac{-dt^2 + dx^2 + dy^2 + dz^2}{d\tau^2} = -1 \quad (\text{interpret.})$$

4-momentum

$$p^\mu = m u^\mu = \underbrace{m\gamma}_{m_{rel}} \frac{dx^\mu}{dt}$$

m_{rel}: relativistic mass (inertia) depends on v

total 4-momentum is conserved

$$\hookrightarrow \frac{dp^\mu}{d\tau} = 0$$

for an isolated body

$\leadsto u^\mu = \text{const}$

$\leadsto$ straight line in spacetime

what is the meaning of p^0? → expand in v

$$p^0 = m\gamma = \frac{m}{\sqrt{1-v^2}} \approx m + \underbrace{\frac{m}{2} v^2}_{\text{kin. energy}} + \dots$$

Rest-mass energy

p^0 is the energy!

Energy and momentum mix under Lorentz trafo!

$$p_\mu p^\mu = m^2 u_\mu u^\mu = -m^2$$

<u>Example</u>: Lorentz force

Faraday differential 2-form F

$$(F_{\alpha\beta}) = \begin{array}{c|c} & \beta \longrightarrow \\ \hline \alpha \downarrow & \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \end{array}$$

Lorentz force: $\dfrac{dp^\mu}{d\tau} = \underset{\uparrow \text{charge}}{q}\, F^\mu{}_\nu\, u^\nu$

this can be checked for each component:

$$\frac{dp^x}{d\tau} = \frac{d(m_{rel} v^x)}{d\tau} = q\left(E_x u^0 + B_z u^2 - B_y u^3\right)$$

$$= q\,\underset{\substack{\| \\ \frac{dt}{d\tau}}}{\gamma}\,\big(E_x + \underbrace{B_z v^y - B_y v^z}_{(\vec{v}\times\vec{B})^x}\big)$$

$$\rightsquigarrow \frac{d(m_{rel} v^x)}{dt} = q\left[E_x + (\vec{v}\times\vec{B})^x\right] \checkmark$$

similar: $\mu \to y$ and $\mu \to z$

What is the meaning of $\mu = 0$? (exercise?)

$$\underset{\substack{\uparrow \\ \text{need to pull} \\ \text{down } \mu;\ p_0 = -p^0}}{-}\frac{d\, m_{rel}}{d\tau} = q\,\underset{\substack{\| \\ \frac{dt}{d\tau}}}{\gamma}\left(-E_x v^x - E_y v^y - E_z v^z\right)$$

$$\rightsquigarrow \underbrace{\frac{d\, m_{rel}}{dt}}_{\approx \frac{d(\frac{1}{2} m v^2)}{dt}} = q\, \vec{v}\cdot\vec{E} = \frac{\text{work done by E-field}}{\text{time interval}}$$

$\approx \dfrac{d(\frac{1}{2}mv^2)}{dt}$ = change of kinetic energy

electric and magnetic field combine into a <u>geometric entity</u> $F_{\alpha\beta}$ on spacetime

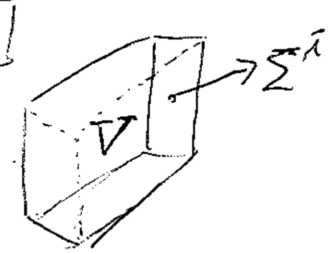
$\rightsquigarrow$ resolves Einsteins problem from lecture 1

(TODO: field equations)

~~charge~~ conservation of charge

$$\frac{\partial}{\partial t}\int dV\cdot\rho = -\int d\Sigma^i\cdot\rho v^i \qquad \boxed{i=1,2,3} \text{ 3-dim. vectors}$$

ρ: charge density; v^i: velocity field of charge element



$$= -\int dV\,\partial_i(\rho v^i) \qquad \boxed{\partial_i \equiv \frac{\partial}{\partial x^i}}$$

$$\rightsquigarrow \frac{\partial \rho}{\partial t} + \partial_i(\rho v^i) = 0 \quad \text{not a tensor expression!}$$

tensor form: $\partial_\mu j^\mu = 0$ with $j^\mu = \begin{pmatrix}\rho \\ \rho v^i\end{pmatrix}$

but is this a 4-vector?

~~vol~~ volume element is contracted in direction of v^i

$\rightsquigarrow \rho = \gamma\rho_0 = \rho_0 u^0$, ρ_0: charge density in rest frame

$\rho v^i = \rho_0\gamma v^i = \rho_0 u^i$

$\rightarrow \boxed{j^\mu = \rho_0\cdot u^\mu}$ $\quad u^\mu$: 4-velocity field of charge elements

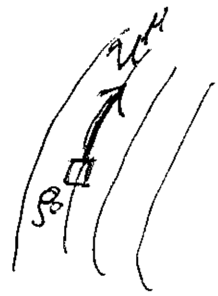
conservation of 4-momentum

conservation of charge $Q \longrightarrow$ current j^μ

conservation of $p^\mu \longrightarrow$ stress-energy tensor $T^{\mu\nu}$ (one index more)

4-momentum density $\rho_0 u^\mu$

ρ_0: energy density in rest-frame of the matter



↳ current of 4-momentum:

$$\rho_0 u^\mu u^\nu = T^{\mu\nu}$$

conservation law:

$$\partial_\nu T^{\mu\nu} = 0$$

explicit:

$$\frac{\partial(\rho_0 u^\mu u^0)}{\partial t} + \partial_i(\rho_0 u^\mu u^i) = 0$$

with $\rho = \gamma^2 \varrho_0$, $u^0 = \gamma$, $u^i = \gamma v^i$

ρ: like ϱ_0, but in moving volume element

$$\rightsquigarrow \frac{\partial(\rho u^\mu)}{\partial t} + \partial_i(\rho u^\mu v^i) = 0$$

conservation of 4-momentum

Remark: $T^{\mu\nu}$ is always symmetric, $T^{\mu\nu} = T^{\nu\mu}$

ideal fluid

above: ~~non-interacting particles~~, dust (all particles moving with same u^μ)

$\rightsquigarrow$ should add pressure p

stress-energy tensor of an ideal fluid:

$$T^{\mu\nu} = \varrho_0 u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)$$

check Newtonian limit:

$\gamma \approx 1$, $u^0 \approx 1$, $u^i \approx v^i$

$$\rightsquigarrow T^{00} = (\varrho_0 + p) u^0 u^0 - p \approx \varrho_0$$

$i, j, k = 1, 2, 3$

$$T^{0j} = T^{j0} = (\varrho_0 + p) u^0 u^j \approx \varrho_0 v^j$$

$$T^{jk} = (\varrho_0 + p) u^j u^k + p\delta^{jk} \approx \varrho_0 v^j v^k + p\delta^{jk}$$

$p = \mathcal{O}(v^2)$!

$(k \to i)$

then: $\boxed{\partial_\mu T^{\nu\mu} = 0}$

$\nu = 0$: $\partial_\mu T^{0\mu} = 0 \rightsquigarrow \boxed{\frac{\partial \varrho_0}{\partial t} + \partial_i(\varrho_0 v^i) = 0}$ equation of continuity

$$\nu = j: \partial_\mu T^{j\mu} = 0 \rightsquigarrow \frac{\partial(\overbrace{\varrho_0 v^j}^{T^{j0}})}{\partial t} + \partial_i(\overbrace{\varrho_0 v^i v^j + \delta^{ji} p}^{T^{ji}}) = 0$$

$$\rightsquigarrow \underbrace{\frac{\partial \varrho_0}{\partial t}}_{-\partial_i(\varrho_0 v^i)} v^j + \varrho_0 \frac{\partial v^j}{\partial t} + \underbrace{\partial_i(\varrho_0 v^i v^j)}_{\partial_i(\varrho_0 v^i) v^j + \varrho_0 v^i \partial_i v^j} + \partial^j p = 0 \quad | : \varrho_0$$

$$\boxed{\frac{\partial v^j}{\partial t} + v^i \partial_i v^j + \frac{1}{\varrho_0} \partial^j p = 0}$$

Euler's equation

$$\rightsquigarrow \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} + \frac{1}{\varrho_0} \vec{\nabla} p = 0$$

> Relativity is not about what is relative, but about what is invariant ! $\rightsquigarrow$ geometry & tensors