curved spacetime, ~~[illegible]~~

Principle(s) of equivalence (PoE)

weak PoE: grav. mass = inertial mass (Galileo, Eötvös)

$\leadsto$ universality of free fall

$\leadsto$ laws of mechanics are the same in a local free falling lab and in the absence of gravity (parabola flight, ISS)

local $\leftrightarrow$ homog. grav. field

Einstein's PoE: laws of mechanics $\to$ laws of physics γ

$\leadsto$ light rays are "falling" (are deflected)

Eddington & Dyson (1919)

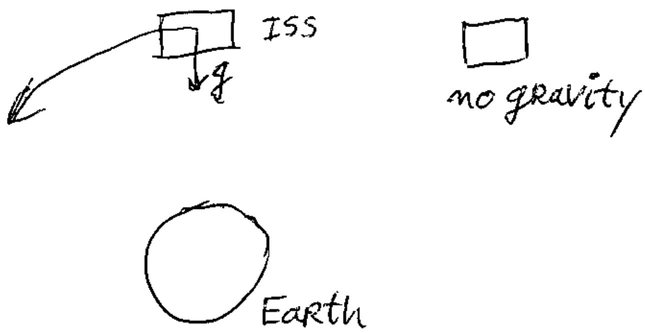
strong PoE: laws of physics include gravity itself

grav. energy contributes equally to inertial and grav. mass

consequences

[PoE: we know laws of Edyn., hydrodyn., thermodyn., ... in presence of gravity]

the following situations are equivalent:



free fall

$\hat{=}$ force free motion

$\hat{=}$ inertial system

$\to$ maximal proper time

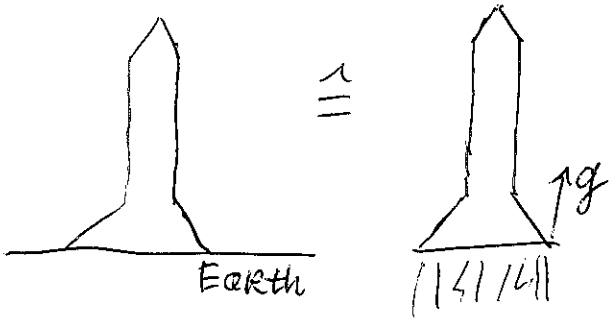
gravitational field

$\sim$ modified/deformed line element

$$d\tau^2 = -g_{\mu\nu}\,dx^\mu dx^\nu \qquad \Delta \to d$$

$\sim$ modified/deformed metric $g_{\mu\nu}$

also equivalent:



gravity experienced on Earth
$\hat{=}$ fictitious force

"the ground is accelerating us upwards"

↳ reduction of proper time (see twin paradox)

↳ time delay in g-field   also: twin par. with a black hole
↳ no acceleration!

<u>thought experiment</u> (maybe skip)

- mass falling in grav. potential from r_0 to r

$$\times r_0 \downarrow \times r \qquad \Delta E = m\left[\phi(r_0) - \phi(r)\right]$$

- convert mass into photon and send it back up

$$\times r_0 \uparrow \times r \qquad E_\gamma = m + \Delta E$$

↑ rest-mass energy $= \underline{\underline{E_{\gamma_0}}}$
= energy arriving back at r_0!
[if conserved]

- energy conservation:

$$\curvearrowright \frac{E_\gamma - E_{\gamma_0}}{E_{\gamma_0}} = \frac{\Delta E}{E_{\gamma_0}}$$

$$E_\gamma = h f$$

$$\curvearrowright \frac{h(f - f_0)}{h f_0} = \frac{\Delta E}{m} = \phi(r_0) - \phi(r)$$

redshift in grav. field

time measurements based on frequencies

↳ time delay!

Differential Geometry IMPRS 2018 L4P3

Physics

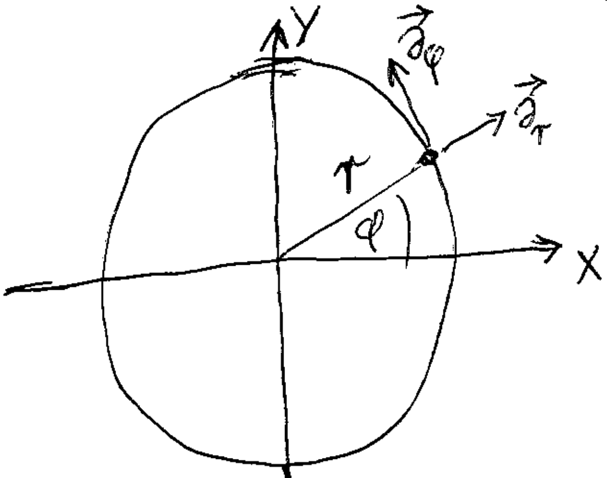
equivalence principle:
special relativistic laws hold in $\underbrace{\text{local free-falling lab}}$
$\leadsto$ coordinate system

Math

principle of general covariance (coordinate invariance)
write these laws of physics in coordinate-invariant form
↳ look at curvilinear coord.

covariant derivative

$\curvearrowright$ example: polar coord.



$x = r\cos\varphi$
$y = r\sin\varphi$ or $\vec{x} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$

$$\vec{\partial}_r = \frac{\partial \vec{x}}{\partial r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$ (i.g. not normalised)

$$\vec{\partial}_\varphi = \frac{\partial \vec{x}}{\partial \varphi} = r\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

line element $ds^2 = dx^2 + dy^2$
$= dr^2 + r^2 d\varphi^2$

general stucture:

$$\vec{\partial}_\mu = \frac{\partial \vec{x}}{\partial x^\mu}$$

x^μ: curvilinear coord. $(x^1 = r,\ x^2 = \varphi)$

$\vec{\partial}_\mu$: curvilin. basis

$$d\vec{x} = \frac{\partial \vec{x}}{\partial x^\mu} \cdot dx^\mu = \vec{\partial}_\mu \cdot dx^\mu$$

$$\curvearrowright ds^2 = d\vec{x} \cdot d\vec{x} = \underbrace{\vec{\partial}_\mu \cdot \vec{\partial}_\nu}_{g(\vec{\partial}_\mu, \vec{\partial}_\nu) = g_{\mu\nu}} dx^\mu dx^\nu$$

spacetime: $d\tau^2 = -ds^2 = -g_{\mu\nu} dx^\mu dx^\nu$

derivative of a vector $\vec{A} = A^\mu \vec{\partial}_\mu$

$$\frac{\partial \vec{A}}{\partial x^\nu} = \frac{\partial A^\mu}{\partial x^\nu}\vec{\partial}_\mu + A^\mu \underbrace{\frac{\partial \vec{\partial}_\mu}{\partial x^\nu}}_{\Gamma^\rho_{\mu\nu}\vec{\partial}_\rho} = \underbrace{\left(\frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\rho\nu} A^\rho\right)}_{\nabla_\nu A^\mu}\vec{\partial}_\mu$$

decompose in basis

Def. of covariant derivative

formula: $\Gamma^\mu_{\rho\nu} = \frac{1}{2} g^{\mu\delta}(\partial_\nu g_{\delta\rho} + \partial_\rho g_{\delta\nu} - \partial_\delta g_{\rho\nu})$, $\quad \partial_\nu \equiv \frac{\partial}{\partial x^\nu}$

$$= \Gamma^\mu_{\nu\rho}$$

Christoffel symbols, formula works in curved spacetime!

back to the example: polar coord. $\mu = r, \varphi$

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$$

only one nonvanishing derivative: $\partial_r g_{\varphi\varphi} = 2r$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{2} r^{-2} \cdot 2r = \frac{1}{r}$$

all other ZERO

$$\Gamma^r_{\varphi\varphi} = -\frac{1}{2} \cdot 2r = -r$$

compare
$$\frac{\partial \vec{\partial}_\varphi}{\partial \varphi} = -r\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix} = -r\vec{\partial}_r$$

$$\frac{\partial \vec{\partial}_\varphi}{\partial r} = \frac{1}{r}\vec{\partial}_\varphi$$

$$\frac{\partial \vec{\partial}_r}{\partial \varphi} = \frac{1}{r}\vec{\partial}_\varphi$$

$$\rightsquigarrow \boxed{\frac{\partial \vec{\partial}_\mu}{\partial x^\nu} = \Gamma^\rho_{\mu\nu}\vec{\partial}_\rho}$$

IMPRS L4P5

properties of ∇_μ: but: $\boxed{\nabla_\mu \nabla_\nu \neq \nabla_\nu \nabla_\mu}$

- like a derivative, e.g. → mandatory

$$\nabla_\mu (A^\nu + B^\nu) = \nabla_\mu A^\nu + \nabla_\mu B^\nu$$

$$\nabla_\mu (F_{\alpha\beta} U^\beta) = (\nabla_\mu F_{\alpha\beta}) U^\beta + F_{\alpha\beta} \nabla_\mu U^\beta$$

- $\nabla_\mu g_{\alpha\beta} = 0 = \nabla_\mu g^{\alpha\beta}$ metric compatibility ↝ Relax: nonmetricity
- $\Gamma^\mu{}_{\rho\nu} = \Gamma^\mu{}_{\nu\rho}$ symmetric ↝ relax: torsion

optional

[math: properties become axioms]

several indices:

$$\nabla_\mu T^\alpha{}_\beta \equiv \partial_\mu T^\alpha{}_\beta + \Gamma^\alpha{}_{\rho\mu} T^\rho{}_\beta - \Gamma^\rho{}_{\beta\mu} T^\alpha{}_\rho$$

upper index: $+\Gamma \ldots$

lower index: $-\Gamma \ldots$

~~an appl~~

meaning of ∇_μ:

$B^\mu \nabla_\mu A^\nu$ is the (covariant) change of a vector $\vec{A}$ along $\vec{B}$