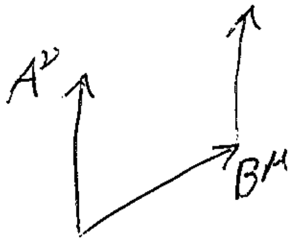
Curved spacetime (continued)

IMPRS2018 L5P1

Recall:

$B^\mu \nabla_\mu A^\nu$ is the (covariant) change of a vector $\vec{A}$ along $\vec{B}$

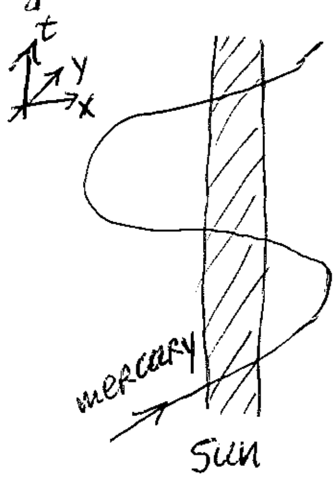
parallel transport of A^ν along B^μ



A^ν does not change along B^μ

$\rightsquigarrow$ $\boxed{B^\mu \nabla_\mu A^\nu = 0}$

geodesics



universality of free fall

$\leftrightsquigarrow$ all masses moving in the same direction of spacetime have the same orbit

↳ geometric description of gravity!

Einstein's idea: free fall in G-field

$\hat{=}$ force-free motion (local)

↳ motion along a straight line in a curved spacetime

↳ geodesic

geodesic = straight line = parallel transport of a vector along itself

[analogy: walking]

↳ $\boxed{u^\nu \nabla_\nu u^\mu = 0}$ geodesic equation

where $u^\mu = \frac{dx^\mu}{d\tau}$ is the tangent to the geodesic

(τ: proper time

u^μ: 4-velocity; $u_\mu u^\mu = -1$)

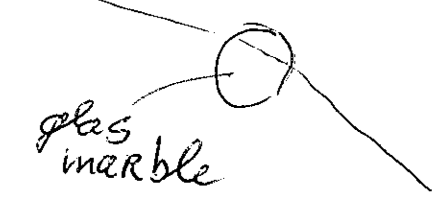
alternative definition of geodesics:

curve of extremal length / proper time

examples:

geodesic on a sphere

optics: light rays take min. optical path

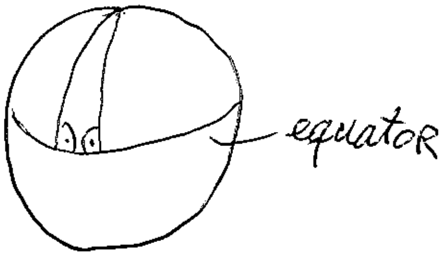


IMPRS LSP2

status: free fall $\rightsquigarrow$ geodesic
gravity $\rightsquigarrow$ metric $g_{\mu\nu}$ (curved spacetime)
$\hookrightarrow$ determines ∇_μ

$\boxed{\text{spacetime tells matter how to move and matter tells spacetime how to curve}}$ J. A. Wheeler

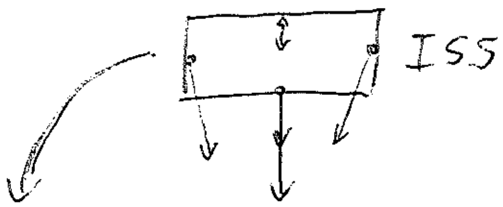
$\hookrightarrow$ need to study <u>curvature</u>



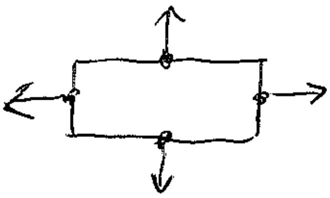
straight lines which are parallel at the equator do not remain parallel
$\Lsh$ geodesic deviation !
also: angles in triangle $\neq 180°$ (Gauss)

<u>geodesic deviation</u>

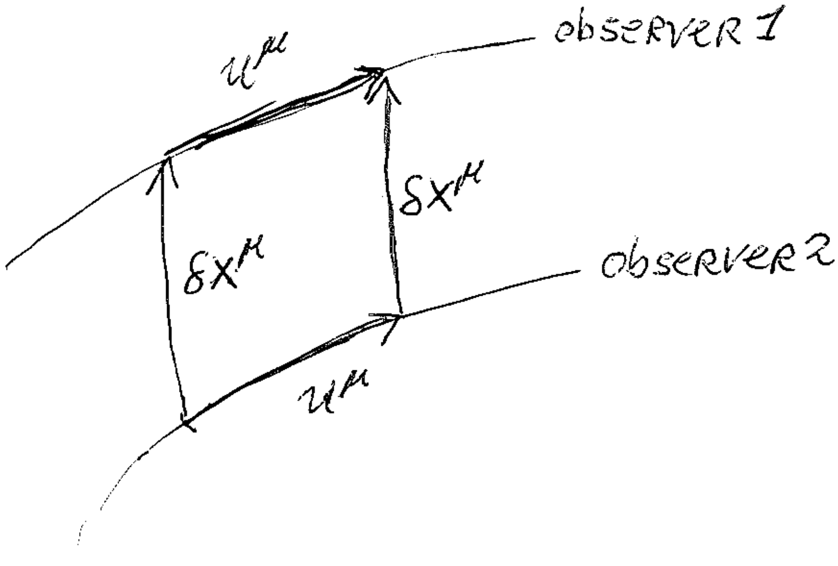
physical connection: tidal forces
nonlocal effect



ISS system:



tidal force = acceleration of two free-falling observers relative to each other



closure of rectangle:
change of u^μ along δx^μ
= " δx^μ " u^μ

$\Lsh$ $\underline{u^\mu \nabla_\mu \delta x^\nu = \delta x^\mu \nabla_\mu u^\nu}$ (*)

geodesics:

$$\underline{u^\nu \nabla_\nu u^\mu = 0} \quad (**)$$

Relative acceleration:

(covariant) chain rule:

$$\frac{D}{D\tau} := \frac{dx^\mu}{d\tau}\nabla_\mu = u^\mu \nabla_\mu$$

$$\frac{D^2 \delta x^\mu}{D\tau^2} = u^\nu \nabla_\nu \big(\underbrace{u^\rho \nabla_\rho \delta x^\mu}_{\delta x^\rho \nabla_\rho u^\mu \quad (*)}\big)$$

Product rule for ∇_ν $\rightarrow$

$$= \underbrace{u^\nu (\nabla_\nu \delta x^\rho)}_{\delta x^\nu \nabla_\nu u^\rho \quad (*)} \nabla_\rho u^\mu + u^\nu \delta x^\rho \nabla_\nu \nabla_\rho u^\mu$$

$$= \underbrace{\delta x^\nu (\nabla_\nu u^\rho) \nabla_\rho u^\mu}_{\nabla_\nu (\underbrace{u^\rho \nabla_\rho u^\mu}_{0 \quad (**)}) - u^\rho \nabla_\nu \nabla_\rho u^\mu, \ \text{prod. rule for } \nabla_\nu} + u^\nu \delta x^\rho \nabla_\nu \nabla_\rho u^\mu$$

$$= -\delta x^\nu u^\rho \nabla_\nu \nabla_\rho u^\mu + u^\nu \delta x^\rho \nabla_\nu \nabla_\rho u^\mu$$

Rename $\nu \leftrightarrow \rho$

$$= u^\nu \delta x^\rho \underbrace{(\nabla_\nu \nabla_\rho - \nabla_\rho \nabla_\nu) u^\mu}_{\text{Def.: } R^\mu{}_{\sigma\nu\rho} u^\sigma + \underbrace{\ldots \nabla u}_{\text{turn out to be zero}}}$$

$$= R^\mu{}_{\sigma\nu\rho} u^\sigma u^\nu \delta x^\rho$$

<u>Definition</u>: Riemann curvature tensor $R^\mu{}_{\sigma\nu\rho}$

$$(\nabla_\nu \nabla_\rho - \nabla_\rho \nabla_\nu) u^\mu = R^\mu{}_{\sigma\nu\rho} u^\sigma$$

after some algebra:

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu{}_{\beta\nu} - \partial_\beta \Gamma^\mu{}_{\alpha\nu} + \Gamma^\mu{}_{\alpha\rho}\Gamma^\rho{}_{\beta\nu} - \Gamma^\mu{}_{\beta\rho}\Gamma^\rho{}_{\alpha\nu}$$

curvature $\rightarrow$ geodesic deviation $\rightarrow$ tidal forces

Remark: one can make Γ vanish locally, but in general not R

$$\boxed{R \text{ is a tensor, but not } \Gamma}$$

definitions:

- Ricci tensor: $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$
- Ricci scalar: $R = R_{\mu\nu} g^{\mu\nu} = R^{\mu}{}_{\mu}$

Symmetries:

$$R_{\mu\nu} = R_{\nu\mu}$$

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\nu\mu}$$

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} = -R_{\mu\nu\beta\alpha}$$

Bianchi identities $\rightarrow$ Jacobi identity $[[\nabla_\alpha, \nabla_\beta], \nabla_\gamma] + \text{cyclic} = 0$

commutator

$$\Updownarrow \quad R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha} = 0 \qquad \text{(1st Bianchi id.)}$$

$$\nabla_\gamma R_{\mu\nu\alpha\beta} + \nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} = 0 \qquad \text{(2nd ")}$$

Outlook: field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \cdot T_{\mu\nu}$$

grav. constant

stress-energy tensor