Exercises – Relativity

1st IMPRS and geo-Q lecture week 2018

Solutions will be provided at: `http://jan-steinhoff.de/lectures/imprs2018/`

1 Truck in the tunnel

A 20 m long truck drives into a 10 m tunnel at a speed v such that $\gamma \equiv (1 - v^2)^{-1/2} = 2$. Due to Lorentz contraction, the truck is only 10 m long in the tunnel system. One could, very briefly, close a barrier at both sides of the tunnel and the truck would fit in perfectly! However, in the truck frame, the truck is 20 m long, while the tunnel is contracted to 5 m. How can a 20 m truck fit in a 5 m tunnel? Relativity theory must be wrong!

1. Refute this claim. For this purpose, draw two spacetime diagrams, one in the tunnel frame and one in the truck frame, containing the worldlines of both ends of truck and tunnel (four worldlines), and the four events where the worldlines cross. Describe in words what these events mean.

2. Discuss what happens if the tunnel ends in a brick wall and a barrier is closed once the truck is inside the tunnel (observed in the tunnel frame).

2 Twin Paradox

Let us discuss the twin paradox from the lecture again using three *inertial* reference frames: the system of Earth, the system of spaceship “Enterprise” flying 4 lightyears from Earth to α Cen in 5 years (as observed by people on Earth), and spaceship “Voyager” flying from α Cen to Earth with the same speed. Neither of the spaceships is accelerating or breaking at any point, they pass by Earth and α Cen with a constant velocity. Furthermore, Enterprise and Voyager fly past α Cen at the same time (but in opposite directions). Let us discuss what happens to a stop watch on Earth, which starts counting when Enterprise speeds past it. You might want to draw spacetime diagrams as a help.

1. When the spaceships are flying past α Cen, how much time passed on the stop watch in the Enterprise frame?

2. When the spaceships are flying past α Cen, how much time passed on the stop watch in the Voyager frame?

3. Discuss this result in the context of relativity of simultaneity.

4. An astronaut jumps from Enterprise to Voyager as both of them are close to α Cen. What can he say about the stop watch on Earth? Roughly discuss the case where he continuously changes his velocity (accelerates) from the Enterprise to the Voyager frame.

3 Einstein and the mirror

Einstein is holding a mirror on his extended arm. He is running very very fast. What does he observe in the mirror if he almost runs at the speed of light?

Now he is falling into a very large (supermassive) black hole, his arm and mirror pointing towards it. What does he observe when the mirror passes through the event horizon (from where no light can escape to infinity)?

4 Christoffel symbol

Obtain the metric and the inverse metric for the surface of a sphere with radius $r = \text{const.}$ Then calculate the Christoffel symbols by

$$\Gamma^{\mu}{}_{\rho\nu} = \frac{1}{2} g^{\mu\delta} \left(\frac{\partial g_{\delta\rho}}{\partial x^{\nu}} + \frac{\partial g_{\delta\nu}}{\partial x^{\rho}} - \frac{\partial g_{\rho\nu}}{\partial x^{\delta}} \right) \tag{1}$$

Hints: Use $x^1 = x^\theta = \theta$ and $x^2 = x^\phi = \phi$ as coordinates on the surface, and

$$\vec{\xi} = r \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} . \tag{2}$$

5 Curvature

Compute the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$, the Ricci tensor $R_{\mu\nu}$, and the Ricci scalar R for the surface of a sphere discussed in the last exercise.

Hint: Due to its symmetries, the Riemann tensor in two dimensions only has one independent nonzero component.