Ehlers 2018 Ex. P1

1.1. Estimates

Remark: your result might vary by some orders of magnitude from the ones below, that's OK :-)

1.) a fist:

$$\mathcal{L} \sim \frac{G}{5c^5} \cdot \frac{(1\,\mathrm{kg})^2 (0.3\,\mathrm{m})^4}{(0.2\,\mathrm{s})^6}$$ with forearm

$$\underline{\underline{\sim 10^{-51}\,\mathrm{W}}} \mathrel{\hat{=}} 10^{-51} \frac{\mathrm{J}}{\hbar (0.2\,\mathrm{s})^{-1}} \cdot \frac{\text{gravitons}}{\mathrm{s}}$$

$\sim 10^{-19}$ gravitons per second

$\boxed{\mathrel{\hat{=}} \text{1 graviton in 100 billion years}}$ ☹

cp. age of universe: 10 "

2.) steel rod:

$$\mathcal{L} \sim \frac{G}{5c^5} \cdot \frac{(5\cdot 10^5\,\mathrm{kg})^2 (20\,\mathrm{m})^4}{(0.1\,\mathrm{s})^6} \sim \boxed{10^{-31}\,\mathrm{W}}$$

⇕ still super small ☹

$(0.1\,\mathrm{s})^6$ ← men moves through system twice per period

3.) Earth:

$$\mathcal{L} \sim \frac{G}{5c^5} \cdot \frac{(6\cdot 10^{24}\,\mathrm{kg})^2 (3\cdot 10^{11}\,\mathrm{m})^4}{(2\cdot 10^7\,\mathrm{s})^6}$$

just Earth, sun not moving; size of system ~ 2AU; $\frac{1}{2}$ year

$\sim 0.1\,\mathrm{W}$ a human produces ~100W of heat ☹

4.) black holes:

$$\mathcal{L} \sim \frac{G}{5c^5} \cdot \frac{(10^{32}\,\mathrm{kg})^2 (5\cdot 10^5\,\mathrm{m})^4}{(0.01\,\mathrm{s})^6}$$

both black holes; $r = \frac{G}{c^2} 6 \cdot 60 M_\odot$

$(0.01\,\mathrm{s})^6$ from Kepler's law $T = 2\pi \sqrt{\frac{r^3}{G\cdot 60 M_\odot}} \sim 0.02\,\mathrm{s}$

$\sim 10^{45}\,\mathrm{W} \mathrel{\hat{=}} \boxed{0.01 \frac{M_\odot c^2}{\mathrm{s}}}$ ⇕ huge ☹

$$\dddot{Q} \sim \sqrt{\frac{5c^5}{G} \cdot \mathcal{L}} \sim 10^{49} \frac{\mathrm{kg\,m^2}}{\mathrm{s}^3}$$

huge power, but still tiny amplitude/strain!

at 1AU:

$$h \sim \frac{G\cdot 2}{c^4} \cdot (0.01\,\mathrm{s}) \cdot \dddot{Q} \cdot \frac{1}{(10^{11}\,\mathrm{m})} \sim \underline{\underline{10^{-8}}} \mathrel{\hat{=}} \frac{\varnothing\,\text{hair}}{10\,\mathrm{km}}$$

⇕ very small

at 10 M:

$$h \sim \frac{G\cdot 2}{c^4} (0.01\,\mathrm{s}) \cdot \dddot{Q} \cdot \frac{1}{10^{23}\,\mathrm{m}} \sim \underline{10^{-20}} \mathrel{\hat{=}} \frac{\varnothing\,\text{proton}}{100\,\mathrm{km}}$$ → measured by LIGO ☹

Ehlers 2018 Ex P2

1.2. binary system (circular)

with $\phi = \omega t$: $\dot{\phi} = \omega$

$$\dot{M}^{11} = -2\mu r^2 \omega \sin\phi\cos\phi, \quad \ddot{M}^{11} = -2\mu r^2\omega^2 \underbrace{(\cos^2\phi - \sin^2\phi)}_{2\cos^2\phi - 1}, \quad \dddot{M}^{11} = \mu r^2 8\omega^3 \sin\phi\cos\phi$$

$$\dot{M}^{22} = 2\mu r^2 \omega \sin\phi\cos\phi = -\dot{M}^{11}, \quad \ddot{M}^{22} = -\ddot{M}^{11}, \quad \dddot{M}^{22} = -\dddot{M}^{11}$$

$$\dot{M}^{12} = \mu r^2 \omega \underbrace{(\cos^2\phi - \sin^2\phi)}_{2\cos^2\phi - 1}, \quad \ddot{M}^{12} = -\mu r^2 4\omega^2 \sin\phi\cos\phi, \quad \dddot{M}^{12} = -\mu r^2 4\omega^3(2\cos^2\phi - 1)$$

Luminosity:

$$\mathcal{L} = +\frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle$$

$$= \frac{1}{5}\langle \dddot{Q}_{ij}(\dddot{M}^{ij} - \frac{1}{3}\delta^{ij}\dddot{M}^{k}{}_{k})\rangle \quad \to 0 \text{ da } \dddot{Q}_{ij}\delta^{ij} = 0$$

$$= \frac{1}{5}\langle \underbrace{\dddot{M}_{ij}\dddot{M}^{ij}}_{(\dddot{M}^{11})^2 + 2(\dddot{M}^{12})^2 + (\dddot{M}^{22})^2} - \frac{1}{3}\underbrace{(\dddot{M}^{k}{}_{k})^2}_{\dddot{M}^{11} + \dddot{M}^{22} = 0}\rangle$$

$$= \frac{1}{5}\underbrace{\int_{\text{orbit}}\frac{dt}{T}}_{\int_0^{2\pi}\frac{d\phi}{2\pi}} \cdot (4\mu r^2\omega^3)^2 \cdot \underbrace{(8\sin^2\phi\cos^2\phi + 2(2\cos^2\phi - 1)^2)}_{8\cos^2\phi(1-\cos^2\phi) + 8\cos^4\phi - 8\cos^2\phi + 2 = 2}$$

$$\boxed{T = \frac{2\pi}{\omega}, \; \phi = \omega t}$$

$$= \frac{1}{5}(4\mu r^2\omega^3)^2 \cdot 2 \cdot \underbrace{\int_0^{2\pi}\frac{d\phi}{2\pi}}_{1}$$

$$= \frac{1}{10}\mu^2 r^4 (2\omega)^6$$

Using: $r = (M\omega^{-2})^{1/3}$

$$\mathcal{L} = \frac{32}{5}\mu^2 M^{4/3}\omega^{10/3}$$

$$= \frac{32}{5}(\underbrace{\mu^{3/5}M^{2/5}}_{M_c}\omega)^{10/3}$$

Compare 1.1

3. Earth:

$M_c = 10^{27}\,kg$, $\omega = \frac{2\pi}{10^7 s}$ ← year

$\leadsto \mathcal{L} \approx 200\,W$

4. black holes:

$M_c = 5\cdot 10^{31}\,kg$, $\omega = \sqrt{\frac{M}{r^3}} = 230\,Hz$

$\leadsto \mathcal{L} \approx 2\cdot 10^{48}\,W \mathrel{\hat{=}} 10\,\frac{M_\odot}{s}$

WOW 8

2.1. Wave equation

Christoffel:

$$2\Gamma^{\mu}_{\alpha\beta} = g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}), \quad g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

$$\partial_\alpha g_{\beta\nu} = \partial_\alpha h_{\beta\nu} = \mathcal{O}(h)$$

$$= (\eta^{\mu\nu} + \mathcal{O}(h))(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}) \quad \blacksquare$$

Curvature:

$$2R_{\mu\nu\alpha\beta} = 2\cdot\Big(\underbrace{g_{\mu\rho}}_{\eta+\mathcal{O}(h)}\underbrace{\partial_\alpha \Gamma^{\rho}_{\beta\nu}}_{\mathcal{O}(h)} - \underbrace{g_{\mu\rho}}_{\eta+\mathcal{O}(h)}\underbrace{\partial_\beta \Gamma^{\rho}_{\alpha\nu}}_{\mathcal{O}(h)} + \underbrace{\Gamma\Gamma - \Gamma\Gamma}_{\mathcal{O}(h^2)}\Big)$$

$$= \underbrace{\partial_\alpha\partial_\beta}_{\text{symmetric in } \alpha,\beta \rightsquigarrow 0} h_{\nu\mu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - (\alpha \leftrightarrow \beta) + \mathcal{O}(h^2)$$

$$= \partial_\alpha\partial_\nu h_{\beta\mu} + \partial_\beta\partial_\mu h_{\alpha\nu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu} + \mathcal{O}(h^2) \quad \blacksquare$$

$$2R_{\mu\nu} = 2R^{\alpha}{}_{\mu\alpha\nu} = \partial^\alpha\partial_\mu h_{\nu\alpha} + \underbrace{\partial_\nu\partial^\alpha h_{\alpha\mu}}_{\partial^\alpha\partial_\nu h_{\mu\alpha}} - \underbrace{\partial^\alpha\partial_\alpha}_{\Box} h_{\mu\nu} - \partial_\mu\partial_\nu \underbrace{h^{\alpha}{}_{\alpha}}_{-\bar{h}^{\alpha}{}_{\alpha}} + \mathcal{O}(h^2)$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \eta_{\mu\nu}\bar{h}^{\alpha}{}_{\alpha}/2 \longrightarrow$$

$$= \partial^\alpha\partial_\mu \bar{h}_{\nu\alpha} - \tfrac{1}{2}\partial_\nu\partial_\mu \bar{h}^{\alpha}{}_{\alpha} + \partial^\alpha\partial_\nu \bar{h}_{\mu\alpha} - \tfrac{1}{2}\partial_\mu\partial_\nu \bar{h}^{\alpha}{}_{\alpha} - \Box\bar{h}_{\mu\nu} + \tfrac{1}{2}\eta_{\mu\nu}\Box\bar{h} + \partial_\mu\partial_\nu \bar{h}^{\alpha}{}_{\alpha}$$

$$= -\Box\bar{h}_{\mu\nu} + \partial^\alpha\partial_\mu \bar{h}_{\nu\alpha} + \partial^\alpha\partial_\nu \bar{h}_{\mu\alpha} + \tfrac{1}{2}\eta_{\mu\nu}\Box\bar{h}^{\alpha}{}_{\alpha} + \mathcal{O}(h^2) \qquad (*)$$

$$2R = 2R^{\mu}{}_{\mu} = -\Box\underbrace{\bar{h}^{\mu}{}_{\mu}}_{\bar{h}^{\alpha}{}_{\alpha}} + \underbrace{\partial^\alpha\partial^\mu \bar{h}_{\mu\alpha} + \partial^\alpha\partial^\mu \bar{h}_{\mu\alpha}}_{\mu\to\beta} + \tfrac{1}{2}\cdot 4\cdot\Box\bar{h}^{\alpha}{}_{\alpha} + \mathcal{O}(h^2)$$

$$= 2\partial^\alpha\partial^\beta \bar{h}_{\alpha\beta} + \Box\bar{h}^{\alpha}{}_{\alpha} \qquad (**)$$

Einstein equations:

$$-2\Big(R_{\mu\nu} - \tfrac{1}{2}\underbrace{g_{\mu\nu}}_{\approx\eta_{\mu\nu}}R\Big) \approx -(*) + \tfrac{1}{2}\eta_{\mu\nu}(**)$$

$$= \Box\bar{h}_{\mu\nu} - \partial^\alpha\partial_\mu h_{\nu\alpha} - \partial^\alpha\partial_\nu h_{\mu\alpha} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} \quad = -16\pi T_{\mu\nu}$$

$\blacksquare$

2.2 Gauge Trafos

$$g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\,,\quad \frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu + \frac{\partial \xi^\alpha}{\partial x'^\mu} \underset{\uparrow}{=} \delta^\alpha_\mu + \frac{\partial \xi^\alpha}{\partial x^\mu} + \mathcal{O}(h^2) \approx \delta^\alpha_\mu + \partial_\mu \xi^\alpha$$

$x = x' + \mathcal{O}(h)$, $\xi = \mathcal{O}(h)$, $\boxed{\partial_\mu = \partial'_\mu + \mathcal{O}(h)}$

$$\eta_{\mu\nu} + h'_{\mu\nu} = (\eta_{\alpha\beta} + h_{\alpha\beta})(\delta^\alpha_\mu + \partial_\mu \xi^\alpha)(\delta^\beta_\nu + \partial_\nu \xi^\beta)$$

$$= \underbrace{\eta_{\alpha\beta}\delta^\alpha_\nu \delta^\beta_\mu}_{\eta_{\mu\nu}} + \underbrace{h_{\alpha\beta}\delta^\alpha_{\mu}\delta^\beta_{\nu}}_{h_{\mu\nu}} + \underbrace{\eta_{\alpha\beta}\partial_\mu \xi^{\alpha}\,\delta^\beta_\nu}_{\partial_\mu \xi_\nu} + \underbrace{\eta_{\alpha\beta}\,\delta^\alpha_\mu\,\partial_\nu \xi^\beta}_{\partial_\nu \xi_\mu} + \mathcal{O}(h^2)$$

$$\Rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \mathcal{O}(h^2)$$

for $\bar{h}_{\mu\nu}$: ~~$h_{\mu\nu} = \bar{h}_{\mu\nu} \frac{1}{2}\eta_{\mu\nu}\bar{h}^\alpha{}_\alpha$~~

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h'^\alpha{}_\alpha$$

$$= h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \tfrac{1}{2}\eta_{\mu\nu}(h^\alpha{}_\alpha + 2\partial_\alpha \xi^\alpha) + \mathcal{O}(h^2)$$

$$= \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\alpha \xi^\alpha + \mathcal{O}(h^2)$$

Now $R'_{\mu\nu\alpha\beta}$:

$$\partial_\nu \partial_\alpha h'_{\mu\beta} = \partial_\nu \partial_\alpha h_{\mu\beta} + \partial_\nu \partial_\alpha \partial_\mu \xi_\beta + \partial_\nu \partial_\alpha \partial_\beta \xi_\mu + \mathcal{O}(h^2)$$

$$\partial_\mu \partial_\beta h'_{\nu\alpha} = \partial_\mu \partial_\beta h_{\nu\alpha} + \partial_\mu \partial_\beta \partial_\nu \xi_\alpha + \partial_\mu \partial_\beta \partial_\alpha \xi_\nu + \mathcal{O}(h^2)$$

$$-\partial_\mu \partial_\alpha h'_{\nu\beta} = -\partial_\mu \partial_\alpha h_{\nu\beta} - \partial_\mu \partial_\alpha \partial_\nu \xi_\beta - \partial_\mu \partial_\alpha \partial_\beta \xi_\nu + \mathcal{O}(h^2)$$

$$-\partial_\nu \partial_\beta h'_{\mu\alpha} = -\partial_\nu \partial_\beta h_{\mu\alpha} - \partial_\nu \partial_\beta \partial_\mu \xi_\alpha - \partial_\nu \partial_\beta \partial_\alpha \xi_\mu + \mathcal{O}(h^2)$$

$\Rightarrow$ all term with ξ cancel $\checkmark$

$$\Rightarrow R'_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \mathcal{O}(h^2)$$

harmonic gauge:

$$0 \overset{!}{=} \partial^\nu \bar{h}'_{\mu\nu} = \underbrace{\partial^\nu \bar{h}_{\mu\nu}}_{f_\mu} + \partial_\mu \partial^\nu \xi_\nu + \underbrace{\partial^\nu \partial_\nu}_{\Box} \xi_\mu \underbrace{- \eta_{\mu\nu}\partial^\nu \partial_\alpha \xi^\alpha}_{-\partial_\mu \partial^\nu \xi^\alpha}$$

$\Rightarrow$ $\Box \xi_\mu = -f_\mu$ $\Leftarrow$ inhom. wave equation for ξ_μ $\Rightarrow$ can (usually) be solved!

3.1. TT-gauge

1.) $\bar{h}_{\mu\nu} = a_{\mu\nu} e^{ik_\alpha x^\alpha} + \text{c.c.}$ then: $\partial_\mu \to i k_\mu$

$\underset{\partial^\alpha \partial_\alpha}{\overset{\shortparallel}{\Box}} \bar{h}_{\mu\nu} = 0 \;\rightsquigarrow\; \underbrace{k_\alpha k^\alpha}_{-k^0k^0 + k^ik^i} = 0 \checkmark \;\rightsquigarrow\; \omega^2 = (k^0)^2 = k^i k_i \quad (*)$

gauge condition: $\partial^\nu \bar{h}_{\mu\nu} = 0 \;\rightsquigarrow\; a_{\mu\nu} k^\nu = 0 \checkmark$

2.) $\Box h^{TT}_{\mu\nu} = \underbrace{\Box \bar{h}_{\mu\nu}}_{0} + \partial_\mu \underbrace{\Box \xi_\nu}_{0} + \partial_\nu \underbrace{\Box \xi_\mu}_{0} - \eta_{\mu\nu} \partial_\alpha \underbrace{\Box \xi^\alpha}_{0} = 0 \checkmark$

$\partial^\nu h^{TT}_{\mu\nu} = \underbrace{\partial^\nu \bar{h}_{\mu\nu}}_{0} + \underline{\partial_\mu \partial^\nu \xi_\nu} + \underbrace{\partial^\nu \partial_\nu \xi_\mu}_{= \Box \xi_\mu = 0} \underbrace{- \eta_{\mu\nu} \partial^\nu \partial_\alpha \xi^\alpha}_{\underline{-\partial_\mu \partial^\nu \xi_\nu}} = 0 \checkmark$

from (17):

$a^{TT}_{\mu\nu} e^{ik\cdot x} + \text{c.c.} = \left(a_{\mu\nu} + ik_\mu \cdot \frac{1}{i\omega} b_\nu + ik_\nu \cdot \frac{1}{i\omega} b_\mu - \eta_{\mu\nu} i k_\alpha \cdot \frac{1}{i\omega} b^\alpha\right) e^{ik\cdot x} + \text{c.c.}$

$\rightsquigarrow a^{TT}_{\mu\nu} = a_{\mu\nu} + n_\mu b_\nu + n_\nu b_\mu - \eta_{\mu\nu} n_\alpha b^\alpha \checkmark$

with $n_\mu = k_\mu / \omega$, $n_0 = \underset{\uparrow_{k_0 = -\omega}}{\frac{k_0}{\omega}} = -1$, $n^i n_i = \frac{k_i k^i}{\omega^2} \overset{(*)}{=} 1$

3.) $a^{TT\,\mu}_{\mu} = 0$ & $a^{TT}_{0i} = 0$ ~~$\rightsquigarrow$ 4 equations~~ in (19)

$\hookrightarrow$ 4 linear equations, can be solved for 4 components of b_μ ✓

from (20): $a^{TT}_{\mu\nu} n^\nu = 0 \xrightarrow{\mu=0} a^{TT}_{0\nu} n^\nu = 0 \rightsquigarrow a^{TT}_{00} \underset{1}{\overset{\shortparallel}{n^0}} + \underbrace{a^{TT}_{0j}}_{0} n^j = 0 \rightsquigarrow a^{TT}_{00} = 0 \checkmark$

$\xrightarrow{\mu=i} a^{TT}_{i\nu} n^\nu = 0 \rightsquigarrow \underbrace{a^{TT}_{i0}}_{0} \underset{1}{\overset{\shortparallel}{n^0}} + a^{TT}_{ij} n^j = 0 \rightsquigarrow a^{TT}_{ij} n^j = 0 \checkmark$

4.) (22) is ~~valid~~ correct, if $\Box h^{TT}_{ij} = 0$ and (21) is fulfilled

$\Box h^{TT}_{ij} = \Lambda_{ij}{}^{kl} \underbrace{\Box \bar{h}_{kl}}_{0} = 0 \checkmark$

$n^i \Lambda_{ik} = \underbrace{n^i \delta_{ik}}_{n_k} - \underbrace{n^i n_i}_{1} n_k = 0 \rightsquigarrow n^i \Lambda_{ij}{}^{kl} = 0 \rightsquigarrow n^i h^{TT}_{ij} = 0 \checkmark$

$\delta^{ij} \Lambda_{ij} = \underbrace{\delta^{ij} \delta_{ij}}_{3} - \underbrace{\delta^{ij} n_i n_j}_{1} = 2 \rightsquigarrow \delta^{ij} \Lambda_{ij}{}^{kl} = \Lambda^{ik} \Lambda_i{}^l - \underbrace{\delta^{ij} \Lambda_{ij}}_{2} \Lambda^{kl} \cdot \frac{1}{2} = 0$

$\rightsquigarrow h^{TT\,i}_{i} = h^{TT\,\alpha}_{\alpha} = 0 \checkmark$

3.1.4.) optional (alternative) part: Ehlers 2018 Ex P6

$$0 = a^{TT\mu}_{\mu} \overset{(19)}{=} a_\mu{}^\mu + n^\mu b_\mu + n^\mu b_\mu - 4 \underbrace{n_\alpha b^\alpha}_{n^\mu b_\mu} = a_\mu{}^\mu - 2 \underbrace{n^\mu b_\mu}_{n^0 b_0 + n^i b_i,\; n^0 = 1} \Rightarrow \boxed{b_0 = \tfrac{1}{2} a_\mu{}^\mu - n^i b_i}$$

$$0 = a^{TT}_{0i} \overset{(19)}{=} a_{0i} + \underset{-1}{n_0} b_i + n_i \underset{\frac{1}{2} a_\mu{}^\mu - n^j b_j}{b_0} \qquad (*)$$

$$\hookrightarrow 0 = n^i (\quad " \quad) = a_{0i} n^i - b_i n^i + \tfrac{1}{2} a_\mu{}^\mu - \underbrace{n^j b_j}_{b_i n^i}$$

$$\Rightarrow b_i n^i = \tfrac{1}{2} a_{0i} n^i + \tfrac{1}{4} a_\mu{}^\mu$$

$$\Rightarrow \boxed{b_0 = \tfrac{1}{4} a_\mu{}^\mu - \tfrac{1}{2} a_{0i} n^i}$$

in $(*)$: $0 = a_{0i} - b_i + n_i \left(\tfrac{1}{4} a_\mu{}^\mu - \tfrac{1}{2} a_{0j} n^j\right)$

$$\Rightarrow \boxed{b_i = a_{0i} - \tfrac{1}{2} a_{0j} n^j n_i + \tfrac{1}{4} n_i a_\mu{}^\mu}$$

harmonic gauge condition for $\bar{h}_{\mu\nu}$: $\bar{h}_{\mu\nu} n^\nu = 0 \Leftrightarrow a_{\mu\nu} n^\nu = 0$

$$\Leftrightarrow \underset{+1}{n^0} a_{\mu 0} + a_{\mu j} n^j = 0 \quad \Leftrightarrow a_{\mu 0} = -a_{\mu j} n^j$$

$$\mu = 0: a_{00} = -\underbrace{a_{0j} n^j}_{-a_{ji} n^i} = a_{ij} n^i n^j$$

and $a_\mu{}^\mu = -a_{00} + a_{ij} \delta^{ij} = (\delta^{ij} - n^i n^j) a_{ij} = \Lambda^{ij} a_{ij}$

then:
$$\boxed{\begin{aligned} b_0 &= \tfrac{1}{2} a_{kl} n^k n^l + \tfrac{1}{4} \Lambda^{kl} a_{kl} \\ b_i &= -a_{ij} n^j + \tfrac{1}{2} a_{kj} n^k n^j n_i + \tfrac{1}{4} n_i \Lambda^{kl} a_{kl} \end{aligned}}$$

insert in eq. (19):

$$\begin{aligned} a^{TT}_{ij} &= a_{ij} + n_i \left(-a_{kj} n^k + \tfrac{1}{2} a_{kl} n^k n^l n_j + \tfrac{1}{4} n_j \Lambda^{kl} a_{kl}\right) \\ &\quad + n_j \left(-a_{il} n^l + \tfrac{1}{2} a_{kl} n^k n^l n_i + \tfrac{1}{4} n_i \Lambda^{kl} a_{kl}\right) \\ &\quad - \delta_{ij} \underbrace{\left(\underset{1}{n^0} b_0 + n^k b_k\right)}_{\frac{1}{2} \Lambda^{kl} a_{kl}} \\ &= a_{ij} - a_{kj} n_i n^k - a_{il} n_j n^l + a_{kl} n^k n^l n_i n_j + \tfrac{1}{2} n_i n_j \Lambda^{kl} a_{kl} - \delta_{ij} \tfrac{1}{2} \Lambda^{kl} a_{kl} \\ &\overset{!}{=} (\delta_i^k - n_i n^k)(\delta_j^l - n_j n^l) a_{kl} - \tfrac{1}{2} (\delta_{ij} - n_i n_j) \Lambda^{kl} a_{kl} \\ &\overset{!}{=} \left(\Lambda_i^k \Lambda_j^l - \tfrac{1}{2} \Lambda_{ij} \Lambda^{kl}\right) a_{kl} \underset{a_{kl} = a_{lk}}{=} \underbrace{\tfrac{1}{2} \left(\Lambda_i^k \Lambda_j^l + \Lambda_i^l \Lambda_j^k - \Lambda_{ij} \Lambda^{kl}\right)}_{\Lambda_{ij}{}^{kl}} a_{kl} \end{aligned}$$

$\Rightarrow (22)$ ▥

3.2 Geodesic Deviation

Ehlers 2018 Ex. P7

choose $r^0=0$ ⇒ r^μ is distance in lab. frame

~~for $h_{\mu\nu}=0$: geodesics at rest stay at rest~~

~~⇒ $(u^\mu)=(1,0,0,0)+O(h)$~~

in lab. frame: $\tau=t$, $(u^\mu)=(1,0,0,0)$, $g_{\mu\nu}\approx\eta_{\mu\nu}$, $\Gamma^\mu_{\alpha\beta}\approx 0$

on worldline $\Gamma^\mu_{\alpha\beta}=0$ ⇒ $\boxed{\frac{d\Gamma^\mu_{\alpha\beta}}{d\tau}=0}$

$$\frac{D^2 r^i}{D\tau}=\frac{d^2r^i}{dt^2}=R^i{}_{\alpha\beta\nu}u^\alpha u^\beta r^\nu=R^i{}_{00j}r^j$$

Use (14): $R^i{}_{00j}=\bar R^i{}_{00j}+O(h^2)$, with $\bar h_{\mu\nu}=h_{\mu\nu}=h^{TT}_{\mu\nu}$, $h^{TT\mu}_{\mu}=0=h^{TT}_{0\mu}$

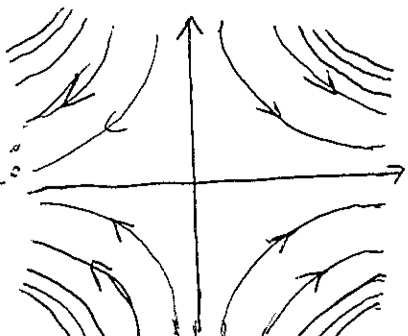
$$\overset{(9)}{=}\frac{1}{2}\left(\partial_0^2 h^{TT i}_j+\partial^i\partial_j\underbrace{h^{TT}_{00}}_{0}-\partial^i\partial_0\underbrace{h^{TT}_{0j}}_{0}-\partial_0\partial_i\underbrace{h^{TT i}_0}_{0}\right)$$

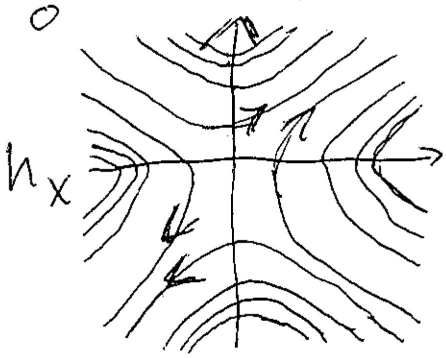
$$\Rightarrow \boxed{\frac{d^2r^i}{dt^2}=\frac{1}{2}r^j\frac{\partial^2 h^{TT i}_j}{\partial t^2}}$$

Force field: $F^i=\frac{1}{2}\partial_0^2 h^{TT i}_j r^j$

$\mathrm{div}\,\vec F=\frac{\partial F^i}{\partial r^i}=\frac{1}{2}\partial_0^2\underbrace{h^{TT i}_j\delta^j_i}_{0}=0$ ⇒ field lines have no source i.e. no start and end

h_+:





oscillation: field lines "vanish" to infinity and come back.

in TT-gauge:

for $h_{\mu\nu}=0$: geodesics at rest stay at rest ⇒ $(u^\mu)=(1,0,0,0)+O(h)$

$\frac{dr^\mu}{d\tau}=0+O(h)$: no initial relative velocity

$$\Rightarrow \frac{D^2r^i}{D\tau^2}=\frac{d^2r^i}{d\tau}+\frac{d}{d\tau}\Big(\underbrace{\Gamma^i{}_{\alpha\beta}u^\alpha r^\beta}_{\frac{1}{2}(\partial_0 h^{TT i}_j+\partial_j \underbrace{h^{TT i}_0}_{0}-\partial_i \underbrace{h^{TT}_{0j}}_{0})\cdot r^j+O(h^2)}\Big)+\underbrace{\Gamma\frac{dr}{d\tau}+\Gamma\Gamma}_{O(h^2)}$$

$\tau=t+O(h)$

$$\approx\frac{d^2r^i}{dt^2}+\frac{1}{2}\partial_0^2 h^{TT i}_j r^j\overset{!}{=}\frac{1}{2}r^j\partial_0^2 h^{TT i}_j$$

$$\Rightarrow \boxed{\frac{d^2r^i}{dt^2}\approx 0}$$

coordinates move along geodesics in TT-gauge!