Gravitational Wave Astronomy

AEI Spring School

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Contents

1 The Quadrupole Formula

(*For typos, suggestions or clarifications please email andrea.antonelli@aei.mpg.de*).
The aim of this exercise is to estimate orders of magnitude associated to different gravitational wave (GW) sources, e.g. the magnitude of the luminosity L and the strains h. We will first estimate two mundane sources of GWs, namely a person waving his/her fist and a steel rod rotating at some frequency. In a second stage, we study binary sources, such as for instance the Earth rotating around the Sun and binary compact objects. The relevant formulae are:

- The luminosity:
$$L = \frac{G}{5c^5} < \dddot{Q}_{ij}\dddot{Q}^{ij} > \tag{1}$$
.

- The strain $h = \frac{\Delta L}{L}$, i.e. the fractional difference in length when GW passes through an object:
$$h \sim \frac{2G\ddot{Q}}{Rc^4} \tag{2}$$

A useful unit converter is given by the link: $\frac{G}{5c^5} \sim 10^{-53}$ W^{-1}. Since we eventually want estimates, we calculate the luminosity as a function of the scales present in the system:

$$L \sim \frac{G}{5c^5} \times \left(\frac{\text{part of mass that moves}^2 \times \text{size of system}^4}{\text{timescale}^6} \right) \sim \frac{G}{5c^5} \times \text{W}^2 \tag{3}$$

1.1 Estimates

Fist moving

A hand might weight around 1 kg and it's around 0.3 m long (*your calculations might different quite considerably depending on your inputs here!*). If we move our fist fast enough, say fast enough to bring it back to equilibrium in a quarter of a second $t \sim 0.25s$, the luminosity of the waves we produce is:

$$L_{fist} \sim \frac{G}{5c^5} \frac{(1\text{kg})^2 \times (0.3\text{m})^4}{(0.25\text{s})^6} \sim 10^{-51}\text{W} \tag{4}$$

This would correspond to 10^{-19} gravitons per second, implying we would have to move constantly in periods of $t \sim 0.25s$ for as long as ∼100 billion years (or 10 times the age of the Universe) to produce a single, lonely graviton. That is arguably not an efficient source!

Steel Rod

Next, consider a 5×10^8 kg steel rod of 20m in length with frequency $\omega = 5$ Hz. The timescale of the problem is dictated by the frequency: $T = \frac{2\pi}{\omega} = 1.25$s. When the quantities are inserted in (3), the following luminosity (or something close to it) should be obtained:

$$L_{rod} \sim 10^{-31}\text{W} \tag{5}$$

Earth rotating around the Sun

Let us work in a reference frame where the Sun is not moving, so that the "part of mass that moves" in the binary is that of the Earth: $M_{Earth} = 6 \times 10^{24}$ kg. The size of the system is the astronomical unit (AU), i.e. $R_{sun-earth} \sim 2 \times 10^{11}$m. The timescale is half a year, 2×10^7s. Plugging in the numbers, one gets:

$$L_{Earth} \sim 0.1\text{W} \tag{6}$$

Clearly, this is much more than we were getting before, e.g. (4) or (5), but is the power created cataclismic? If you have hit the gym recently, chances are that in one hour exercising you produced 100W of power: this means that half a year of the Earth rotation produces 1/1000 of what humans can produce in everyday life activities in a much shorter timescale. One could argue that, given the vicinity to the source, we could still see GW from the Earth-Sun binary using observatories much less sensitive than LIGO. But is this a sensible argument? (Un)fortunately, we cannot see GW produced in the *near zone* of the GW, e.g. within the wavelength of the wave λ_{GW}. The physical picture is very intuitive: in the near zone, one cannot distinguish the perturbation h_{ab} from the curvature R_{ab} produced by the source. It is simply not possible to disentangle the two! Only when the wave has left the near zone of any source and entered the *far zone* one could reliably distinguish the background, which would now be simply flar Minkowski η_{ab}, from the perturbation. For this reason the favourite sources of GW astronomers are compact objects, i.e. system so far away that our observatories would comfortably be in the far zone and yet so compact to produce cataclismic events.

Binary BHs

The BHs are taken to have $\mathrm{M}_{BH} \sim 30\mathrm{M}_{Sun}$ ($\mathrm{M}_{tot} = 60\mathrm{M}_{Sun}$), as for instance in GW150914. They are taken to be close to the ISCO:

$$R_{ISCO} \sim 6\frac{\mathrm{GM}_{tot}}{c^2} \sim 5 \times 10^5\mathrm{m} \tag{7}$$

The timescale is given by Kepler's 3rd law:

$$\mathrm{T} = 2\pi\sqrt{\frac{R^3_{ISCO}}{\mathrm{GM}_{tot}}} \sim 0.01\mathrm{s} \tag{8}$$

The luminosity is calculated in the same way as before:

$$L_{BH-BH} \sim \frac{G}{5c^5}\frac{(10^{32}\mathrm{kg})^2 \times (5 \times 10^5\mathrm{m})^4}{(0.01\mathrm{s})^6} \sim 10^{45}\mathrm{W} \sim 0.01\frac{\mathrm{M}_{sun}\mathrm{c}^2}{\mathrm{s}} \tag{9}$$

Compared to the the Earth-Sun binary system, we see that the BH BH is insanely huger in terms of power. The difference, as can be appreciated from (9) and (6) is a combination of much bigger mass in the BH binary and, mostly, from the shorter timescale associated to the latter system. How does this huge generated power translate to an observable strain? The Quadrupole is first found via the luminosity, eqn.(1):

$$\dddot{Q} \sim \sqrt{\frac{5c^5L}{\mathrm{G}}} \sim 10^{49}\frac{\mathrm{kgm}^2}{\mathrm{s}^3} \tag{10}$$

We can estimate the strain from (2). At 1AU$\sim 10^{11}$m[1] we'd get:

$$h_{1AU} \sim 10^{-8} \sim \frac{10^{-4}\mathrm{m}}{10\mathrm{km}} \sim \frac{\text{width of a hair}}{10\mathrm{km}} \tag{11}$$

(to obtain the result, we multiplied by the timescale t$\sim$ 0.01s). The LIGO sources observed so far are more or less D$\sim$ 0.5Mpc away. This corresponds to a strain:

$$h_{1/2Mpc} \sim 10^{-20} \sim \frac{10^{-15}\mathrm{m}}{100\mathrm{km}} \sim \frac{\text{size of proton}}{100\mathrm{km}} \sim \frac{10^{-4}\mathrm{m}}{10^{16}\mathrm{m}} \sim \frac{\text{size of hair}}{\text{distance Earth to TRAPPIST-1}} \tag{12}$$

[1]Don't forget we would not be able to measure something so close, given we would be deep into the near zone. Honestly, observing waves so close would be the least of problems; I would first worry about the tidal forces ripping Earth apart.

1.2 Binary System on a circular orbit

Now onto a more realistic source. Binary systems in circular orbits are exactly what we would expect in LIGO/VIRGO [2]. Here we'll use the formulae:

$$Q^{ij} = M^{ij} - \frac{1}{3}M^k_k\delta^{ij} \tag{13}$$
$$M^{11} = \mu r^2 \cos^2 \omega t \tag{14}$$
$$M^{22} = \mu r^2 \sin^2 \omega t \tag{15}$$
$$M^{12} = M^{21} = \mu r^2 \cos\omega t \sin\omega t \tag{16}$$

Using $\phi = \omega t$ (so that $\dot{\phi} = \omega$) and taking three time derivatives of (14-16), you should be able to obtain:

$$\dddot{M}^{11} = 8\mu r^2\omega^3 \sin\phi\cos\phi \tag{17}$$
$$\dddot{M}^{22} = -\dddot{M}^{11} \tag{18}$$
$$\dddot{M}^{12} - 4\mu r^2\omega^3(2\cos^2\phi - 1) \tag{19}$$

where the relation $\cos^2\phi - \sin^2\phi = 2\cos^2\phi - 1$ has been used to polish the results. The luminosity is calculated as follows:

$$L = \frac{1}{5} < \dddot{Q}_{jk}\dddot{Q}^{jk} >= \frac{1}{5} < \dddot{Q}_{jk}\left(\dddot{M}^{jk} - \frac{1}{3}\dddot{M}\delta^{jk}\right) >= \frac{1}{5} < \dddot{Q}_{jk}\dddot{M}^{jk} - \frac{1}{3}\dddot{Q}_{jk}\dddot{M}\delta^{jk} >$$
$$= \{\text{Use that quadrupole is traceless, e.g. } \dddot{Q}_{jk}\delta^{jk} = 0\} = \frac{1}{5} < \dddot{Q}_{jk}\dddot{M}^{jk} >$$
$$= \frac{1}{5} < \left(\dddot{M}_{jk} - \frac{1}{3}\dddot{M}\delta_{jk}\right)\dddot{M}^{jk} >= \frac{1}{5} < \dddot{M}_{jk}\dddot{M}^{jk} - \frac{1}{3}\dddot{M}\delta_{jk}\dddot{M}^{jk} >= \frac{1}{5} < \dddot{M}_{jk}\dddot{M}^{jk} - \frac{1}{3}\dddot{M}^2 >$$
$$= \frac{1}{5} < (\dddot{M}^{11})^2 + 2(\dddot{M}^{12})^2 + (\dddot{M}^{22})^2 - \frac{1}{3}[\dddot{M}^{11} + \dddot{M}^{22}] >= \{\text{From (18), use that } \dddot{M}^{11} = -\dddot{M}^{22}\}$$
$$= \frac{1}{5} < (\dddot{M}^{11})^2 + 2(\dddot{M}^{12})^2 + (\dddot{M}^{22})^2 >$$

We then take the average $< .. >$ over an orbit:

$$\begin{aligned}
L =& \frac{1}{5}\int_0^{2\pi}\frac{d\phi}{2\pi}[(\dddot{M}^{11})^2 + 2(\dddot{M}^{12})^2 + (\dddot{M}^{22})^2] \\
& \frac{1}{5}\int_0^{2\pi}\frac{d\phi}{2\pi}(4\mu r^2\omega^3)^2\left[8\sin^2\phi\cos^2\phi + 2(2\cos^2\phi - 1)^2\right] \\
&= \frac{1}{5}\int_0^{2\pi}\frac{d\phi}{2\pi}(4\mu r^2\omega^3)^2\left[8\cos^2\phi(1-\cos^2\phi) + 8\cos^4\phi - 8\cos^2\phi + 2\right] \\
&= \frac{1}{5}\int_0^{2\pi}\frac{d\phi}{2\pi}(4\mu r^2\omega^3)^2[2] = \frac{\mu r^4}{10}(2\omega)^6 = \frac{32}{5}\mu^2\mathrm{M}_{tot}^{4/3}\omega^{10/3}
\end{aligned}$$

[2] The circularization of the orbit due to GW, i.e. one of the most important results in GW theory, has a profound effect on LIGO/VIRGO science. Circular orbits are much, much easier to treat than eccentric ones and they allow a reliable modelling of the source, which is crucial for matched filtering.

In the last line we have related $r(\omega)$ via Kepler's third law: $r = (M_{tot}\omega^{-2})^{1/3}$. We can rearrange the mass quantities in the last line into a single mass parameter, the **chirp mass**.

$$L = \frac{32}{5}(\mu^{3/5}M_{tot}^{2/5}\omega)^{10/3} = \frac{32}{5}(\mathcal{M}_{chirp}\omega)^{10/3} \tag{20}$$

The Sun-Earth binary has a chirp mass: $\mathcal{M}_{chirp}^{Sun-Earth} = 10^{27}$kg, which together with $\omega \sim 2\pi$/yr corresponds to a luminosity of $L \sim 200$W. The BH-BH system has a chirp mass of $\mathcal{M}_{chirp}^{BHBH} = 5 \times 10^{31}$kg and a frequency from Kepler's law of $\omega \sim 230$Hz; the associated luminosity is:

$$L_{BHBH} \sim 10^{49}\text{W} \tag{21}$$

This corresponds to a luminosity in the GW spectrum that (during the BHs mergers) is bigger than the luminosity of all the galaxies in the universe combined in the Electromagnetic spectrum (!!).

2 Linearised Gravity

In this exercise, we'll learn how to linearise the full GR equations in order to make them more tractable. We will also introduce the gauge symmetry of GR.

2.1 The full linearised wave equation

First, we calculate the Christoffel symbols and the Riemann curvature in linearised gravity, by employing the usual metric perturbation: $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$.

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}) \tag{22}$$

$$= \frac{1}{2}(\eta^{\mu\nu} + O(h))(\partial_\alpha(\eta_{\beta\nu} + h_{\beta\nu}) + \partial_\beta(\eta_{\alpha\nu} + h_{\alpha\nu}) - \partial_\nu(\eta_{\alpha\beta} + h_{\alpha\beta})) \tag{23}$$

$$= \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}) \tag{24}$$

$$R^{\mu}_{\nu\alpha\beta} = \partial_\alpha\Gamma^{\mu}_{\nu\beta} - \partial_\beta\Gamma^{\mu}_{\nu\alpha} + \Gamma\Gamma - \Gamma\Gamma = \{\text{Use that } \Gamma\Gamma \sim O(h^2)\} \tag{25}$$

$$= \partial_\alpha(\frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\beta\sigma} + \partial_\beta h_{\nu\sigma} - \partial_\sigma h_{\nu\beta})) - \partial_\beta(\frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\alpha\sigma} + \partial_\alpha h_{\nu\sigma} - \partial_\sigma h_{\nu\alpha})) \tag{26}$$

$$= \frac{1}{2}\big[\eta^{\mu\sigma}\partial_\alpha\partial_\nu h_{\beta\sigma} + \eta^{\mu\sigma}\partial_\alpha\partial_\beta h_{\nu\sigma} - \eta^{\mu\sigma}\partial_\alpha\partial_\sigma h_{\nu\beta} - (\alpha \leftrightarrow \beta)\big] \tag{27}$$

$$= \frac{1}{2}\big[\partial_\alpha\partial_\nu h^{\mu}_{\beta} - \partial_\alpha\partial^\mu h_{\nu\beta} - (\alpha \leftrightarrow \beta)\big] \Rightarrow 2R_{\mu\nu\alpha\beta} = \partial_\alpha\partial_\nu h_{\mu\beta} - \partial_\mu\partial_\alpha h_{\nu\beta} - (\alpha \leftrightarrow \beta)\big] \tag{28}$$

Where in the last line we used the symmetry in α and β of the second term to cancel it and we manipulated the indices. We need two more pieces of equipment in order to linearise the

Einstein's equations, namely the Ricci tensor and scalar. In what follows, we'll find it useful to use a redefinition of the perturbation[3]:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{29}$$

$$\bar{h} = h - \frac{4}{2}h = -h \tag{30}$$

The Ricci tensor is:

$$\begin{aligned}
2R_{\mu\nu} = 2R^{\alpha}_{\mu\alpha\nu} &= \partial_\mu\partial^\alpha h_{\nu\alpha} + \partial_\nu\partial^\alpha h_{\alpha\mu} - \partial_\alpha\partial^\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h^{\alpha}_{\alpha} && (31)\\
&= \partial_\mu\partial^\alpha h_{\nu\alpha} + \partial_\nu\partial^\alpha h_{\alpha\mu} - \Box h_{\mu\nu} + \partial_\mu\partial_\nu \bar{h} && (32)\\
&= \partial_\mu\partial^\alpha \bar{h}_{\nu\alpha} - \frac{1}{2}\partial_\mu\partial_\nu\bar{h} + \partial_\nu\partial^\alpha\bar{h}_{\alpha\mu} - \frac{1}{2}\partial_\nu\partial_\mu\bar{h} - \Box\bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box\bar{h} + \partial_\mu\partial_\nu\bar{h} && (33)\\
&= \partial_\mu\partial^\alpha \bar{h}_{\nu\alpha} + \partial_\nu\partial^\alpha\bar{h}_{\alpha\mu} - \Box\bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box\bar{h} && (34)
\end{aligned}$$

In (32), we used the D'Alembertian symbol $\Box = \partial_\alpha\partial^\alpha$ (valid in flat space, as in linearised gravity indices are always raised and lowered using $\eta_{\mu\nu}$) and (30). In (34) we used the symmetry in ν and μ of the partial derivatives. From (34), we can readily obtain the Ricci scalar:

$$2R = 2R^{\mu}_{\mu} = \Box\bar{h} + 2\partial_\alpha\partial^\mu\bar{h}_{\mu\alpha} \tag{35}$$

We plug (34) and (35) in the Einstein's equations to finally obtain the full (linearised) wave equation:

$$\begin{aligned}
&R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}\\
&\Rightarrow \Box\bar{h}_{\mu\nu} - \partial^\alpha\partial_\mu\bar{h}_{\nu\alpha} - \partial^\alpha\partial_\nu\bar{h}_{\mu\alpha} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu} && (36)
\end{aligned}$$

2.2 Gauge Transformations

Equation (36) is not yet ready to be used, as it contains a hidden gauge symmetry that manifests itself under transformations of the form:

$$x^\mu = x'^\mu + \xi^\mu(x') \tag{37}$$

To dig out any physical observable, we eventually want to be able to treat this gauge freedom and remove it. The first step in solving any problem, however, is acknowledging that there is one: here, we study the symmetry that will be removed in Ex.3.
Let us start from calculating the behaviour of the perturbations under (37), calculating the linearisation of the covariance of the metric:

$$g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \tag{38}$$

[3]There is no black magic behind these redefinitions, they are just introduced to make the final gauged equation more visually pleasent and easier to treat (that is, similar to the equation of a wave in flat space).

$$\eta_{\mu\nu} + h'_{\mu\nu} = (\eta_{\alpha\beta} + h_{\alpha\beta})\frac{\partial}{\partial x^\mu}(x^\alpha + \xi^\alpha)\frac{\partial}{\partial x^\nu}(x^\beta + \xi^\beta) \tag{39}$$
$$= (\eta_{\alpha\beta} + h_{\alpha\beta})(\delta^\alpha_\mu + \partial_\mu \xi^\alpha)(\delta^\beta_\nu + \partial_\nu \xi^\beta) \tag{40}$$
$$= \eta_{\alpha\beta}\delta^\alpha_\mu\delta^\beta_\nu + h_{\alpha\beta}\delta^\alpha_\mu\delta^\beta_\nu + \eta_{\alpha\beta}\partial_\mu\xi^\alpha\partial_\nu\xi^\beta + h_{\alpha\beta}\partial_\mu\xi^\alpha\partial_\nu\xi^\beta \tag{41}$$
$$= \eta_{\mu\nu} + h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \tag{42}$$
$$\Rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \tag{43}$$

It does not matter if we choose as done in (39) to work with ∂_μ rather than the more correct ∂'_μ: the difference between the two is order $O(h)$ and it becomes an order $O(h^2)$ when multiplied by $\xi \sim O(h)$. We then calculate the gauge transformation of $\bar{h}_{\mu\nu}$:

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h' \tag{44}$$
$$= h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \frac{1}{2}\eta_{\mu\nu}(h + 2\partial_\alpha\xi^\alpha) + O(h^2) \tag{45}$$

It is also easy (albeit a bit lengthy in algebra) to check that the Riemann curvature is also invariant:

$$2R'_{\mu\nu\alpha\beta} = \partial_\nu\partial_\alpha h'_{\mu\beta} + \partial_\mu\partial_\beta h'_{\nu\alpha} - \partial_\mu\partial_\alpha h'_{\nu\beta} - \partial_\nu\partial_\beta h'_{\mu\alpha} \tag{46}$$
$$\begin{aligned} &= \partial_\nu\partial_\alpha h_{\mu\beta} + \underbrace{\partial_\nu\partial_\alpha\partial_\mu\xi_\beta}_{A} + \underbrace{\partial_\nu\partial_\alpha\partial_\beta\xi_\mu}_{B} + \partial_\mu\partial_\beta h_{\nu\alpha} + \underbrace{\partial_\mu\partial_\beta\partial_\nu\xi_\alpha}_{C} + \underbrace{\partial_\mu\partial_\beta\partial_\alpha\xi_\nu}_{D} \\ &- \partial_\mu\partial_\alpha h_{\nu\beta} - \underbrace{\partial_\mu\partial_\alpha\partial_\nu\xi_\beta}_{A} - \underbrace{\partial_\mu\partial_\alpha\partial_\beta\xi_\nu}_{D} - \partial_\nu\partial_\beta h_{\mu\alpha} - \underbrace{\partial_\nu\partial_\beta\partial_\mu\xi_\alpha}_{C} - \underbrace{\partial_\nu\partial_\beta\partial_\alpha\xi_\mu}_{B} \end{aligned} \tag{47}$$
$$= \partial_\nu\partial_\alpha h_{\mu\beta} + \partial_\mu\partial_\beta h_{\nu\alpha} - \partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\nu\partial_\beta h_{\mu\alpha} + O(h^2) = 2R_{\mu\nu\alpha\beta} + O(h^2) \tag{48}$$

We have a clear symmetry at linear order in h under the (gauge) transformation (37).

3 Plane Waves

Now that we have got hands on dealing with the symmetry of GR, we can start thinking of how to get rid of the spurious degrees of freedom associated with it. Let us first stare at the linearised E.E., equation (36): we notice that there is a recurring theme in all but the first terms in the LHS. We can in fact see that all terms are proportional to the structure $\sim \partial\bar{h}$. It would be incredibly nice to get rid of all those terms by employing the freedom we have just studied. In short, we would want to set up a condition:

$$\partial^\nu \bar{h}_{\mu\nu} = 0 \rightarrow \text{(Lorentz gauge)} \tag{49}$$

Can we do that? Imposing (49), (45) implies that:

$$\partial^\nu \bar{h}_{\mu\nu} = \partial^\nu \bar{h}'_{\mu\nu} + \Box\xi_\mu \tag{50}$$

We would have to solve $\partial^\nu \bar{h}'_{\mu\nu} = -\Box\xi_\mu$ in order for the Lorentz gauge to be feasible. Luckily, the D'Alembertian is invertible, so that the equation always admits solutions. After imposing the Lorentz gauge, the wave equation (36) is given in the far zone ($T_{\mu\nu}$=0) by:

$$\Box\bar{h}_{\mu\nu} = 0 \tag{51}$$

This is not yet the full story! In the next subsection, it will be shown how (50) comes about and that the wave solutions to (51) possess a residual symmetry that we can turn in our favour.

3.1 Tansverse-Traceless (TT) gauge

1) First, let us prove that (51) admits solutions of the type:

$$\bar{h}_{\mu\nu} = a_{\mu\nu} e^{ik_\alpha x^\alpha} + (c.c.) \tag{52}$$

under a particular set of conditions for k_α. Plugging in (52) in (51) and using the magical operator $\partial_\mu \to ik_\mu$, we readily get:

$$\Box \bar{h}_{\mu\nu} = \partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = 0 \to (-k^2)\bar{h}_{\mu\nu} = 0 \Rightarrow k_\alpha k^\alpha = 0 \text{ (Dispersion relation)} \tag{53}$$

$$\partial^\nu \bar{h}_{\mu\nu} = 0 \to k^\nu \bar{h}_{\mu\nu} = k^\nu a_{\mu\nu} e^{ik_\alpha x^\alpha} = 0 \Rightarrow k^\nu a_{\mu\nu} = 0 \text{ (Transverse condition)} \tag{54}$$

2) Now, let us study the (final) symmetry we can play around with. Let us go back for a moment to equation (50), implying that:

$$\Box \xi_\mu = f_\mu \tag{55}$$

where we have set $\partial^\nu \bar{h}'_{\mu\nu} = -f_\mu$. One solution to (55) could be $f_\mu = 0$, corresponding to a choice in which the Lorentz gauge is valid even for the transformed perturbation. We show that such a choice leads to a transformed perturbation $h^{TT}_{\mu\nu}$ that indeed satisfies wave equation and Lorentz condition. Let us begin with equation (45) when the Lorentz gauge is imposed:

$$h^{TT}_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\alpha \xi^\alpha \tag{56}$$

This readily satisfies the given conditions due to the commutation of partial derivatives:

$$\Box h^{TT}_{\mu\nu} = \underbrace{\Box h_{\mu\nu}}_{\text{0,equation (51)}} + \partial_\mu \underbrace{\Box \xi_\nu}_{\text{0, TT gauge choice}} + \partial_\nu \overbrace{\Box \xi_\mu}^{\text{0, TT gauge choice}} - \eta_{\mu\nu}\partial_\alpha \underbrace{\Box \xi^\alpha}_{\text{0, TT gauge choice}} = 0 \tag{57}$$

$$\partial^\nu \bar{h}^{TT}_{\mu\nu} = \underbrace{\partial^\nu \bar{h}_{\mu\nu}}_{\text{0, eqn (49)}} + \partial_\mu \partial^\nu \xi_\nu + \underbrace{\partial_\nu \partial^\nu \xi_\mu}_{\Box \xi^\mu = 0} - \underbrace{\eta_{\mu\nu}\partial^\nu \partial_\alpha \xi^\alpha}_{(\alpha \leftrightarrow \nu) = \partial_\mu \partial^\nu \xi_\nu} = 0 \tag{58}$$

Being the above conditions valid, the TT-gauged perturbation admits wave solutions too:

$$h^{TT}_{\mu\nu} = a^{TT}_{\mu\nu} e^{ik_\alpha k^\alpha} \tag{59}$$

$$\xi_\mu = \frac{1}{i\omega} b_\mu e^{ik_\alpha k^\alpha} \tag{60}$$

Equations (52) and (59) are intimately linked through a condition on their amplitudes. This condition is very important, as it will soon give us the geometric interpretation of GWs in the TT gauge. The condition is easily checked to be:

$$
\begin{aligned}
h^{TT}_{\mu\nu} &= h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\alpha \xi^\alpha && (61)\\
&= a_{\mu\nu} e^{ik_\alpha x^\alpha} + \partial_\mu\left(\frac{1}{i\omega} b_\nu e^{ik_\alpha k^\alpha}\right) + \partial_\nu\left(\frac{1}{i\omega} b_\mu e^{ik_\alpha k^\alpha}\right) - \eta_{\mu\nu}\partial_\alpha\left(\frac{1}{i\omega} b^\alpha e^{ik_\sigma k^\sigma}\right) && (62)\\
&= a_{\mu\nu} e^{ik_\alpha x^\alpha} + \frac{1}{i\omega} b_\nu i k_\mu e^{ik_\alpha k^\alpha} + \frac{1}{i\omega} b_\mu i k_\nu e^{ik_\alpha k^\alpha} - \eta_{\mu\nu}\frac{1}{i\omega} b^\alpha i k_\alpha e^{ik_\sigma k^\sigma} && (63)\\
&= (a_{\mu\nu} + n_\mu b_\nu + n_\nu b_\mu - \eta_{\mu\nu} n^\alpha b_\alpha) e^{ik_\sigma x^\sigma} = 0 && (64)\\
&\Rightarrow a_{\mu\nu} + n_\mu b_\nu + n_\nu b_\mu - \eta_{\mu\nu} n^\alpha b_\alpha = 0 && (65)
\end{aligned}
$$

In line (64), $n_\mu = k_\mu/\omega$ has been defined as an orthonormal basis (it is easily checked that $n_i n^i = 1$ and $n_0 = -1$). Using the same techniques one also quickly checks that:

$$
\partial^\nu h^{TT}_{\mu\nu} = 0 = \partial^\nu a^{TT}_{\mu\nu} e^{ik_\alpha x^\alpha} \rightarrow i k^\nu a^{TT}_{\mu\nu} = i\omega\left(\frac{k^\nu}{\omega}\right) a^{TT}_{\mu\nu} = 0 \Rightarrow n^\nu a^{TT}_{\mu\nu} = 0 \qquad (66)
$$

3) Let us now go back to the condition $\Box \xi_\mu = 0$ and, armed with the machinery developed in this exercise, argue why this choice leaves us with a transverse-traceless (TT) condition. The Einstein's equations have 10 independent variables, which are reduced to 6 independent ones when the wave equation (51) is imposed. If we further choose $\Box \xi_\mu = 0$, we are free to choose 4 conditions and reduce the number of independent variables for the linearised E.E. to only 2. What conditions are ideal to have a good geometric picture of the problem? Consider $a^{TT} = a^{TT}_{0i} = 0$ and (66):

$$
\begin{aligned}
\mu = 0 &\rightarrow a^{TT}_{0\nu} n^\nu = a^{TT}_{00} \underbrace{n^0}_{1} + \underbrace{\overset{TT}{a_{0i}}}_{0} n^i = a^{TT}_{00} = 0 && (67)\\
\mu = i &\rightarrow a^{TT}_{i\nu} n^\nu = \underbrace{a^{TT}_{i0}}_{0} n^0 + a^{TT}_{ij} n^j = a^{TT}_{ij} n^j = 0 && (68)
\end{aligned}
$$

These choices correspond, via (59), to the following conditions on the perturbation: $h^{TT} = 0$, $h^{TT}_{0i} = 0$, $h^{TT}_{00} = 0$ and $h^{TT}_{ij} n^j = 0$. We see that the waves are both traceless and transverse, hence the name we've been using all along! We can then have a picture of the waves as being represented by a matrix which is traceless, transerve and that contains 2 degrees of freedom (or polarizations) h_+ and $h_\times$:

$$
h^{TT}_{ij}(x^\mu) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} e^{ik_\alpha x^\alpha} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos[\omega(t - z/c)] + \text{ imaginary part} \qquad (69)
$$

where in the last passage, the real part of the wave has been taken to move along the z axis in the transverse x-y plane, with velocity equal that of light (according to the wave equation (51)). We then have a nice geometrical picture for the wave if we consider the associted metric:

$$
\begin{aligned}
ds^2 = &- c^2 dt^2 + dz^2 + (1 + h_+ \cos[\omega(t - z/c)]) dx^2 \\
&+ (1 - h_+ \cos[\omega(t - z/c)]) dy^2 + 2h_\times \cos[\omega(t - z/c)] dx dy \qquad (70)
\end{aligned}
$$

With not too much effort, one can visualise the (simultaneous) squeezing and stretching of test masses when hit by a GW starting directly from the above metric.

4) Here we argue that it is possible to project any perturbation $\bar{h}_{\mu\nu}$ onto the TT gauge via the projector $\Lambda_{ij}^{kl} = \frac{1}{2}(\Lambda_i^k \Lambda_j^l + \Lambda_i^l \Lambda_j^k - \Lambda_{ij}\Lambda^{kl})$

$$h_{\mu\nu}^{TT} = \Lambda_{ij}^{kl}\bar{h}_{\mu\nu} \tag{71}$$

In order for this operator to give us the wave solution in the TT gauge, both the transverse condition (68) and the traceless one (66) must be satisfied. Moreover, the wave equation must be satisfied. The latter requirement is satisfied by construction:

$$\Box h_{\mu\nu}^{TT} = \Lambda_{ij}^{kl} \underbrace{\Box \bar{h}_{\mu\nu}}_{0,\text{ eqn (51)}} = 0 \tag{72}$$

To satisfy (66), we show that $\Lambda_{ik} = \delta_{ik} - n_i n_k$ is transverse:

$$n^i \Lambda_{ik} = n^i(\delta_{ik} - n_i n_k) = n_k - \underbrace{n_i n^i}_{1} n_k = 0 \Rightarrow n^i \Lambda_{ik}^{jl} = 0 \Rightarrow n^i h_{ik}^{TT} = 0 \tag{73}$$

whereas to satisfy (68), we resort to the tracelessness of the Lambda tensor:

$$\delta^{ij}\Lambda_{ij} = \underbrace{\delta^{ij}\delta_{ik}}_{3} - \underbrace{\delta^{ij} n_i n_k}_{1} = 2 \tag{74}$$

$$\Rightarrow \delta^{ij}\Lambda_{ij}^{kl} = \frac{1}{2}(\delta^{ij}\Lambda_i^k\Lambda_j^l + \delta^{ij}\Lambda_i^l\Lambda_j^k - \delta^{ij}\Lambda_{ij}\Lambda^{kl}) = \frac{1}{2}(2\Lambda^{ik}\Lambda_i^l - 2\Lambda^{kl}) = 0 \tag{75}$$

$$\Rightarrow h^{TT} = 0 \tag{76}$$

3.2 Geodesic deviation

In this last exercise, we connect the mathematical construct built so far to experiments: the metric (70) tells us that a set of test masses $m_i = 1$ (or, more appropriately for LIGO, a set of rigid rulers) will move according to a + and × pattern dictated by the 2 independent polarizations of the wave. In this last exercise, we want to eventually build up force lines for these polarizations. What do we mean here by "force"? Force is an aleatory concept in General Relativity, however, quite remarkably, it is possible to think of the perturbation of spacetime during the passage of a gravitational wave as a Newtonian force on the test masses. This result requires looking at the problem from a specific frame, usually called the "proper detector frame". In this coordinates, in which close to the observer one has $g_{\mu\nu} \approx \eta_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$, the geodesic deviation,

$$\frac{D^2 r^\mu}{D\tau^2} = R^\mu_{\alpha\beta\nu} u^\alpha u^\beta r^\nu \tag{77}$$

takes a very simple form with:

$$u^\alpha = (1,0,0,0) \tag{78}$$

$$dt^2 = d\tau^2\left[1+\frac{1}{c^2}\frac{dx^i}{d\tau}\frac{dx^i}{d\tau}\right] = d\tau^2[1+O(h^2)] \Rightarrow dt = d\tau \tag{79}$$

$$D_\mu = \partial_\mu + O(h^2) \tag{80}$$

$$\Rightarrow \frac{D^2 r^i}{D\tau^2} \approx \frac{d^2 r^i}{d\tau^2} = R^i_{\alpha\beta\nu}u^\alpha u^\beta r^\nu = R^i_{00j}r^j \tag{81}$$

From (28), we obtain:

$$2R^i_{00j} = \partial_0^2 h_j^{i,TT} + \partial_j\partial^i h_{00}^{TT} - \partial_0\partial^i h_{0j}^{TT} - \partial_0\partial_j h_0^{i,TT} = \partial_0^2 h_j^{i,TT} = \ddot{h}_j^{\ i,TT} \tag{82}$$

$$\Rightarrow \frac{d^2 r_i}{d\tau^2} = \frac{1}{2}\ddot{h}_{ij}^{\ TT} r^j \tag{83}$$

Equation (83) looks very Newtonian, with an acceleration on the LHS and a (unit) force on the RHS. Hence, in the proper detector frame we can think of the RHS in the geodesic deviation as a force on the test of masses. The force field lines can be found through the divergence of the "force":

$$F^i = \frac{1}{2}\ddot{h}_j^{\ i,TT} r^j \Rightarrow \partial_i F^i = \frac{1}{2}\ddot{h}_j^{\ i,TT}\underbrace{\partial_i r^j}_{\delta_i^j} = \frac{1}{2}\ddot{h}^{TT} = 0 \tag{84}$$

The force lines do not have no source and they look as in figure (a) and (b). One can notice the + polarisation in the former and the × one in the latter. We ask a final question: what physical meaning do the TT coordinates described above carry? In order to see it, the geodesic deviation must be calculated with TT information:

$$\frac{D^2 r^i}{D\tau^2} = \frac{d^2 r^i}{d\tau^2} + \frac{d}{d\tau}(\Gamma^i_{\alpha\beta}u^\alpha r^\beta) + O(h^2) \tag{85}$$

$$\approx \frac{d^2 r^i}{dt^2} + \frac{1}{2}\ddot{h}_j^{i,TT} r^j \tag{86}$$

We have to compare (86) to (83):

$$\frac{d^2 r^i}{dt^2} + \frac{1}{2}\ddot{h}_j^{i,TT} r^j = \frac{1}{2}\ddot{h}_j^{i,TT} r^j$$

$$\frac{d^2 r^i}{dt^2} = 0 \tag{87}$$

Equation (87) is telling us that in the TT gauge, the coordinates have no acceleration with respect to the geodesic, i.e. they move along with it!

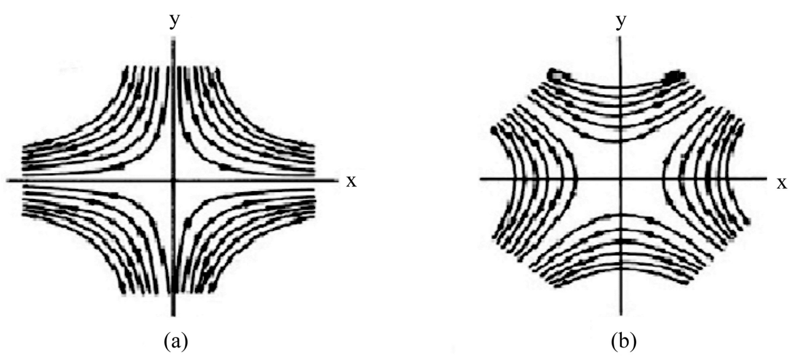

Figure 1: Taken from the internet, sorry for the bad quality