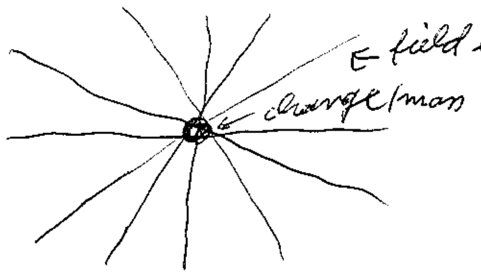
Ehlers 2018 L1P1

Intro, $\mu,\nu=0,1,2,3$, $i,j=1,2,3$, $\eta=\mathrm{diag}(-1,1,1,1)$

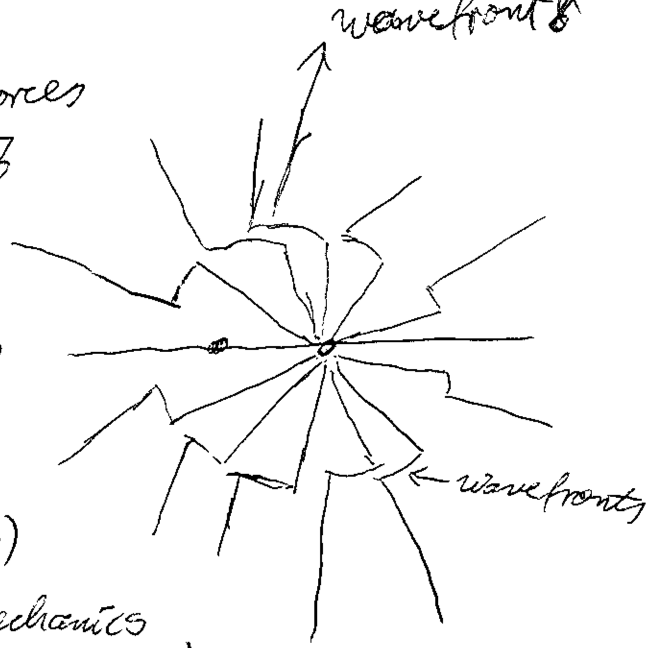
Gravitational waves (GW)

~~in linearized gravity~~

max. speed: c $\Rightarrow$ no instantaneous forces
$\Rightarrow$ propagation, waves!



accelerating charge/mass $\longrightarrow$



(J.J. Thompson)

particle physics: relativity + quantum mechanics
forces $\Rightarrow$ bosonic particles ~ waves (duality)
graviton: massless spin-2 particle
massless particles $\Rightarrow$ 2 polarisations!

linearised gravity

$g_{\mu\nu}\approx\underbrace{\eta_{\mu\nu}}_{\text{flat spacetime}}+h_{\mu\nu}$, $|h_{\mu\nu}|\ll 1$, $\boxed{\text{indices now pulled with } \eta!}$

in $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi T_{\mu\nu}$ $\qquad G=1$

$\Rightarrow \underbrace{\partial_\alpha\partial^\alpha}_{\Box}\bar h_{\mu\nu}-\partial^\alpha\partial_\mu\bar h_{\nu\alpha}-\partial^\alpha\partial_\nu\bar h_{\mu\alpha}+\eta_{\mu\nu}\partial^\alpha\partial^\beta\bar h_{\alpha\beta}=-16\pi T_{\mu\nu}$ (exercise)

where $\bar h_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h$ trace-reversed

gauge/coordinate fixing: $\partial^\alpha\bar h_{\mu\alpha}=0$ harmonic gauge

$$\Rightarrow \boxed{\Box\bar h_{\mu\nu}=-16\pi T_{\mu\nu}}$$

gravity (linear)	electrodynamics
harm. gauge $\partial^\alpha\bar h_{\mu\alpha}=0$	Lorenz gauge $\partial^\alpha A_\alpha=0$
Einstein eq.: $\Box\bar h_{\mu\nu}=-16\pi T_{\mu\nu}$	Maxwell eq.: $\Box A^\mu=-4\pi j^\mu$
solution? $\bar h_{\mu\nu}=4\int d^3x'\,\frac{T_{\mu\nu}(\vec{x}',t_{ret})}{\lvert\vec{x}-\vec{x}'\rvert}$, $t_{ret}=t-\lvert\vec{x}-\vec{x}'\rvert$	$A^\mu=\int d^3x'\,\frac{j^\mu(\vec{x}',t_{ret})}{\lvert\vec{x}-\vec{x}'\rvert}$
quadrupolar radiation	dipolar radiation
2 polarisations $h_+,h_\times$ $\Leftrightarrow$ transverse-traceless waves	2 polarisations $\Leftrightarrow$ transverse waves

more

vacuum $T_{\mu\nu}=0$ ⇝ plane-wave solutions, monochromatic

transverse-traceless (TT) gauge for plane waves: ⇒ makes polarisations / physical DOF manifest

Ehlers 2018 L1P2

e.g. wave in z-direction (exercise)

$$(h^{TT}_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_+ & a_\times & 0 \\ 0 & a_\times & -a_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{a^{TT}_{ij}} e^{ik_\mu x^\mu} + \text{c.c.} \qquad (*)$$

k_μ: wave vector

in general: $h^{TT}_{0\mu}=0$, $\partial^j h^{TT}_{ij}=0$, $\mathrm{Tr}(h)=0$ ⇝ traceless

$k^j a^{TT}_{ij}=0$ ⇝ transverse

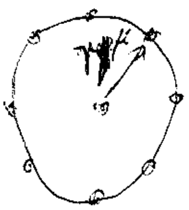
Are GWs observable?

↳ might be a gauge effect

should carry energy, but grav. energy can not be localised (due to equivalence principle)

(also: are there "exakt" GW solutions?)

look at free-falling ring of test-masses:



geodesic deviation w.r.t. center

$$\frac{D^2 r^\mu}{D\tau^2} = R^\mu{}_{\alpha\beta\nu} u^\alpha u^\beta r^\nu$$

τ: proper time

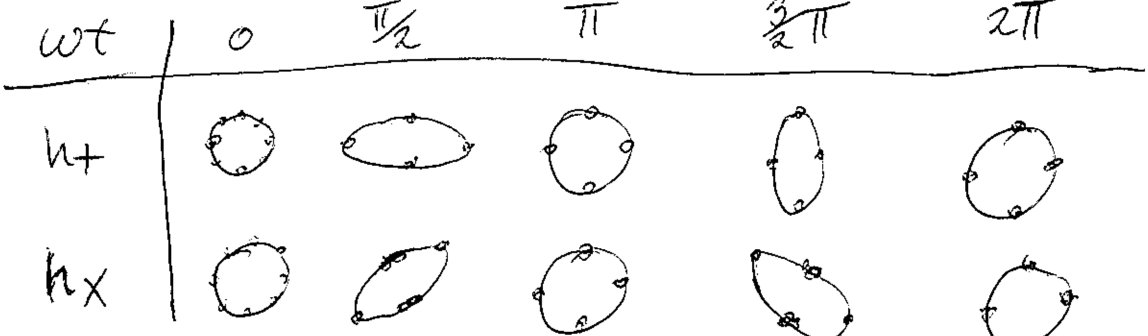
t: coordinate time

in exercise: $\dfrac{d^2 r_i}{dt^2} \approx \dfrac{1}{2} r^j \dfrac{\partial^2 h^{TT}_{ij}}{\partial t^2}$

ansatz: $r^i = r^i_0 + \Delta r^i$, $|\Delta \vec{r}| \ll |\vec{r}_0|$ and $(*)$

$$\Rightarrow \Delta r^i \approx \frac{1}{2} r^j_0 h^{TT}_{ij} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{r}_0$$

$\boxed{h_{+,\times} \sim \sin \omega t}$



circular polarization from superposing h_+ & $h_\times$

GW detectable!

h ~ (dimensionless) strain

[comparison Edyn here]

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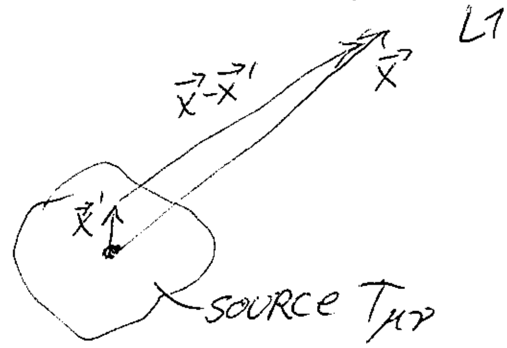
L1P3



far-zone approximation: $|\vec x| \gg |\vec x'|$

$\Rightarrow |\vec x - \vec x'| \approx |\vec x| =: R$, $\boxed{t_{ret} \approx t - R}$

$\Rightarrow \bar h_{\mu\nu} \approx \frac{4}{R} \int d^3x'\, T_{\mu\nu}(\vec x', t_{ret})$

use conservation of energy-momentum:

$$\partial_\nu T^{\mu\nu} = 0 \Rightarrow \partial_i T^{\mu i} = -\dot T^{\mu 0} \quad (*) \qquad \text{here } \dot{} = \partial_0$$

and $\partial_k \partial_l (x^i x^j) = \partial_k (x^i \delta^j_l + \delta^i_l x^j) = \delta^i_k \delta^j_l + \delta^i_l \delta^j_k \quad (**)$

then: $\int d^3x'\, T^{ij} \underset{\uparrow\ T^{ij} = \frac12 (T^{ij} + T^{ji})}{=} \int d^3x\, T^{kl} \cdot \frac12 (\delta^i_k \delta^j_l + \delta^j_k \delta^i_l) \underset{\uparrow\ (**)}{=} \frac12 \int d^3x\, T^{kl}\, \partial_k \partial_l (x^i x^j)$

$$\underset{P.I.}{=} \int d^3x\, \partial_k \partial_l T^{kl} x^i x^j \underset{\uparrow\ (*)}{=} \frac12 \int d^3x\, \ddot T^{00} x^i x^j = \frac12 \left(\frac{d^2}{dt^2} \int d^3x\, T^{00} x^i x^j \right)$$

$\Rightarrow \boxed{\bar h^{ij} \approx \frac{2}{R} \ddot M^{ij}(t_{ret})}$ where $M^{ij} = \int d^3x\, T^{00} x^i x^j$

↳ energy/mass density

compare quadropole: $\cancel{Q^{ij} = \int d^3x}$

$$Q^{ij} = \int d^3x\, T^{00} \left(x^i x^j - \frac13 \delta^{ij} x^k x_k \right)$$

$$= M^{ij} - \frac13 M^k{}_k \delta^{ij}$$

without proof: $\boxed{h^{TT}_{ij} \approx \frac{2}{R} \underbrace{\Lambda_{ij,kl}}_{\text{"TT-projector"}} \ddot Q^{kl}(t_{ret})}$ quadrupole approximation

in TT-gauge: ~~remove longitudinal part~~

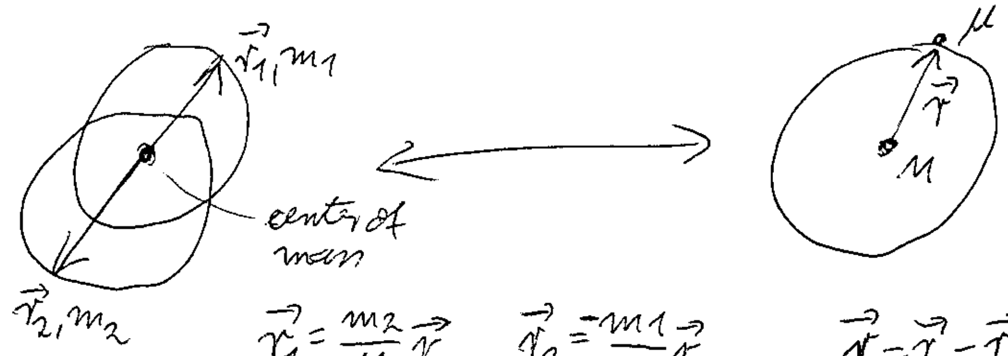
~~& make traceless~~

at observer: ~~looks like~~ a plane wave (R large)

~~choose wave in~~

example: binary system in circular orbit — Ehlers 2018, LTP4

Newtonian estimate



$$\mu = \frac{m_1 m_2}{M}$$

$$M = m_1 + m_2$$

$$\vec{r}_1 = \frac{m_2}{M}\vec{r}, \quad \vec{r}_2 = \frac{-m_1}{M}\vec{r} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2 = r\begin{pmatrix}\cos\phi \\ \sin\phi \\ 0\end{pmatrix}$$

here: circular orbit, $\phi = \omega t$, $r = \text{const}$

point-mass: $T^{00} \approx m_1 \delta(\vec{X} - \vec{r}_1) + m_2 \delta(\vec{X} - \vec{r}_2)$

$$\rightsquigarrow \underline{\underline{M^{ij}}} = m_1 r_1^i r_1^j + m_2 r_2^i r_2^j$$

$$= m_1 \left(\frac{m_2}{M}\right)^2 r^i r^j + m_2 \left(\frac{m_1}{M}\right)^2 r^i r^j$$

$$= \frac{m_1 m_2 (m_2 + m_1)}{M^2} r^i r^j = \underline{\underline{\mu r^i r^j}}$$

in components:

$$M^{11} = \mu r^2 \cos^2\omega t = \tfrac{1}{2}\mu r^2 (1 + \cos 2\omega t)$$

$$M^{22} = \mu r^2 \sin^2\omega t = \tfrac{1}{2}\mu r^2 (1 - \cos 2\omega t)$$

$$M^{12} = M^{21} = \mu r^2 \sin\omega t \cos\omega t = \tfrac{1}{2}\mu r^2 \sin 2\omega t \qquad \text{other zero}$$

$\rightsquigarrow$ $\boxed{\text{GW frequency} = 2 \times \text{orbital frequency !}}$

~~now $h^{ij} \approx \frac{2}{R}\ddot{M}^{ij}$~~

~~in TT-gauge: remove longitudinal~~ polarization

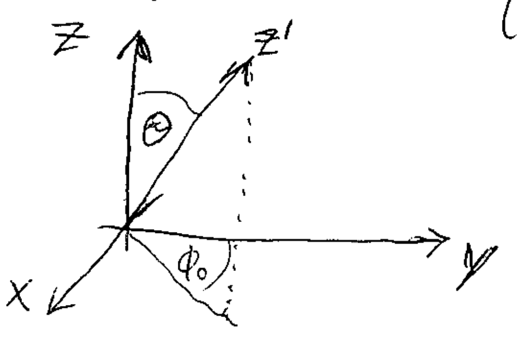
~~& make traceless~~

choose rotated system such that wave is in z-direction at detector

↳ primes M'^{ij}, ...

explicitly: $M'^{ij} = R^i{}_k R^j{}_l M^{kl}$

with the rotation $(R^i{}_k) = \begin{pmatrix}\text{rotation of} \\ \text{yz-plane by } \theta\end{pmatrix}\begin{pmatrix}\text{Rotation of} \\ \text{xy-plane by } \phi_0\end{pmatrix}$



$$= \begin{pmatrix}1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1\end{pmatrix}$$

Then: $\left(h'^{TT}_{ij}\right) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $h_+, h_\times$: polarizations
physical information in GW

How do we get from $\bar{h}'^{ij} \approx \frac{2}{R}\ddot{M}'^{ij}$ to h^{TT}_{ij}?

$\Updownarrow$ remove longitudinal components & make traceless

$$\left(h'^{TT}_{ij}\right) = \left(\bar{h}'_{ij}\right)_{xy\text{-part}} - \text{trace}$$
$$= \frac{2}{R}\begin{pmatrix} \ddot{M}'_{11} & \ddot{M}'_{12} & 0 \\ \ddot{M}'_{12} & \ddot{M}'_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \text{trace} = \frac{2}{R}\begin{pmatrix} (\ddot{M}'_{11}-\ddot{M}'_{22})/2 & \ddot{M}'_{12} & 0 \\ \ddot{M}'_{12} & -(\ddot{M}'_{11}-\ddot{M}'_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Updownarrow \quad \boxed{h_+ = \frac{1}{R}\left(\ddot{M}'_{11} - \ddot{M}'_{22}\right), \quad h_\times = \frac{2}{R}\ddot{M}'_{12}}$$

after some algebra (Mathematica):

$$\boxed{\begin{aligned} h_+ &= -\frac{4\mu\omega^2 r^2}{R}\,\frac{1+\cos^2\Theta}{2}\cos(2\omega t + 2\Phi_0) \\ h_\times &= \underbrace{-\frac{4\mu\omega^2 r^2}{R}}_{\text{amplitude}}\,\underbrace{\cos\Theta}_{\text{angular pattern}}\,\cdot\underbrace{\sin(2\omega t + 2\Phi_0)}_{\text{phase}} \end{aligned}}$$

<u>chirp mass:</u>

~~prefact~~ amplitude factor $\mu\omega^2 r^2$ & 3rd Kepler $\omega^2 r^3 = M$

$\hookrightarrow \mu\omega^2 r^2 = \mu\omega^2\left(\frac{M}{\omega^2}\right)^{2/3} = \mu M^{2/3}\omega^{2/3} \stackrel{!}{=} M_c^{5/3}\omega^{2/3}$

with $\boxed{M_c = \mu^{3/5}M^{2/5} \text{ chirp mass!}}$

combination of masses that determines GW ~~leading~~
(at "leading order" & in early phase)