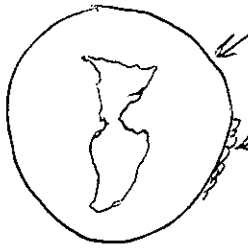
Gravitational waves on curved background

Ehlers 2018 L2P1

~~Beispiel~~: Example: Waves on the ocean



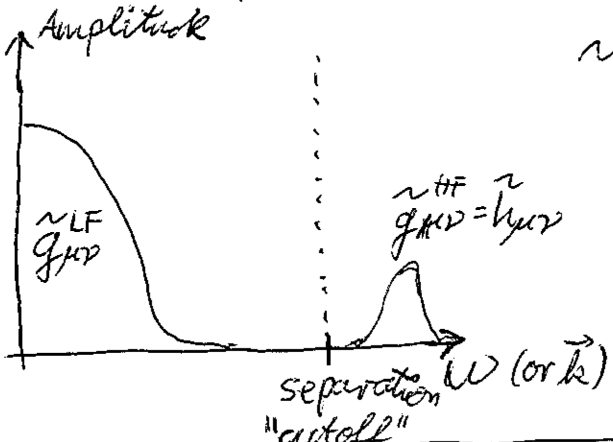
which "part" of the water is the ocean and which one is the wave?
⇝ approximate notion
1.) small amplitude
2.) small wavelength / high frequency

gravitational waves:

decompose $g_{\mu\nu} = \underbrace{g^{LF}_{\mu\nu}}_{\langle g_{\mu\nu}\rangle \text{ average}} + \underbrace{g^{HF}_{\mu\nu}}_{h_{\mu\nu}}$

LF/HF: low/high frequency projections (not unique, problems with covariance; in practice, often obvious)

Assumption: ~~separation of scales~~



$\sim$: Fourier-~~trafo~~

$\Rightarrow \boxed{\tilde g^{LF}_{\mu\nu}\, \tilde h_{\mu\nu} \approx 0}$

(approximate) decoupling ✓

$\boxed{\text{separation of scales} \Rightarrow \text{decoupling}}$

decomposition of field equations, schematically:

$$R = T \rightsquigarrow R^{LF} + h + h^2 + O(h^3) = T^{LF} + T^{HF} + \ldots$$

$\xrightarrow{HF}$ $\Box h = T^{HF} + O(h^2)$

$\xrightarrow[\text{projection}]{LF}$ $R^{LF} = \underbrace{T^{LF}}_{\langle T_{\mu\nu}\rangle} + \underbrace{(h^2)^{LF}}_{\langle h^2\rangle =: T^{GW}_{\mu\nu}} + O(h^3)$

no $O(h)$ term! ⇝ decoupling

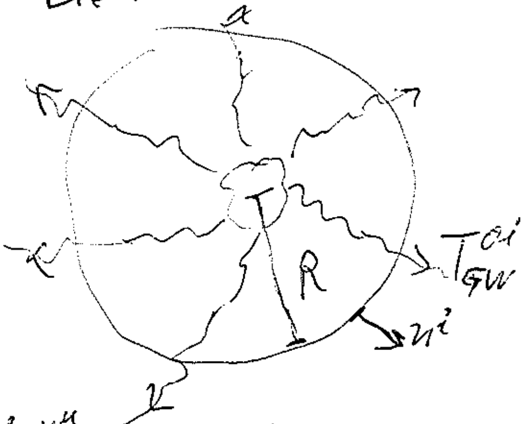
derivation: see Flanaghan, Hughes

for $\langle T_{\mu\nu}\rangle = 0$, $\langle g_{\mu\nu}\rangle \simeq \eta_{\mu\nu} + O(h^2)$

and $h_{\mu\nu}$ in TT-gauge:

$$\boxed{T^{GW}_{\mu\nu} = \frac{1}{32\pi}\langle \partial_\mu h^{TT}_{ij}\, \partial_\nu h^{ij}_{TT}\rangle} \sim (\text{amplitude})^2$$

GW carry energy & momentum!



Quadrupole formula

radiated power $\sim \frac{\text{energy loss}}{\text{time}}$

$\sim$ luminosity $\mathcal{L}$

Energy-flow density: T^{0i}_{GW}

$$\leadsto \mathcal{L} = \oint d\Omega \cdot R^2 n^i \underbrace{T^{0i}_{GW}}$$

$$-T^{GW}_{0i} = \frac{1}{32\pi}\langle \partial_0 h^{TT}_{kl} \partial_i h^{TT}_{kl} \rangle$$

approx. plane wave for $R \to \infty$, $h^{TT} \sim e^{ik_\mu x^\mu}$, $(k_\mu) = (-\omega, k^i) \approx \omega(-1, n^i)$, $\omega^2 = \vec{k}^2$

$$\leadsto \partial_i h^{TT}_{kl} \approx i k_i h^{TT}_{kl} \approx i\omega n_i h^{TT}_{kl}$$

$$\approx -n_i \dot{h}^{TT}_{kl}$$

$$= \frac{1}{8} R^2 \oint \frac{d\Omega}{4\pi} \langle \dot{h}^{TT}_{kl} \dot{h}^{TT}_{kl} \rangle, \quad h^{TT}_{ij} = \Lambda_{ijkl} \frac{2}{R} \ddot{Q}^{kl}(t_{ret})$$

Λ_{ijkl}: dep. on n^i; $\ddot{Q}^{kl}$: indep of $d\Omega$!

$$= \frac{1}{2} \langle \dddot{Q}_{ij} \dddot{Q}_{kl} \rangle \underbrace{\oint \frac{d\Omega}{4\pi} \Lambda_{ijkl}}_{\text{look it up}}$$

$$\boxed{\mathcal{L} = \frac{1}{5} \langle \dddot{Q}_{ij} \dddot{Q}^{ij} \rangle}$$ quadrupole formula

for a binary, circular orbit:

$$\boxed{\mathcal{L} = \frac{32}{5} (\mathcal{M}_c \omega)^{10/3}}$$, exercise 1.2

$\mathcal{M}_c$: chirp mass

energy loss $\to$ binary separation shrinks $\leadsto$ $\boxed{\text{inspiral}}$

energy balance: $$\frac{dE}{dt} = -\mathcal{L} \quad (*)$$

$\frac{dE}{dt}$: $r \neq$ const; $\mathcal{L}$: $r =$ const } adiabatic approximation. separation of timescales: orbit $\ll$ radiation reaction

$$E = E_{kin} + E_{pot} = \frac{1}{2}\mu v^2 - \frac{\mu M}{r} = -\frac{1}{2}\mu\omega^2 R^2 = -\frac{1}{2}\mathcal{M}_c^{5/3}\omega^{2/3}$$

$$v^2 = \omega^2 R^2 = \frac{M}{R}$$ 3rd Kepler

in $(*)$: $\dot{\omega} \cdot \omega^{-11/3} = \frac{32}{5} \cdot 3 \cdot \mathcal{M}_c^{5/3} = \text{const}$

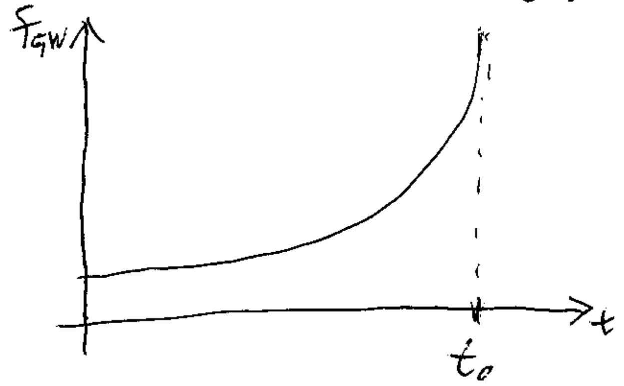
ω increases $\leadsto$ inspiral, ending in a merger!

$\omega \to \infty$ at merger

integrate:
$$\omega^{-8/3} = \frac{256}{5} \mathcal{M}_c^{5/3} \underbrace{\left(-t + t_0\right)}_{\tau\text{: time to merger}}$$

GW frequency $\omega_{GW} = 2\omega = 2\pi f_{GW}$

$$\rightsquigarrow f_{GW} \simeq 134\,\text{Hz} \left(\frac{1.21 M_\odot}{\mathcal{M}_c}\right)^{5/8} \left(\frac{1s}{\tau}\right)^{3/8}$$



frequency evolution $\rightsquigarrow \mathcal{M}_c$

GW150914: $\mathcal{M}_c \approx 30 M_\odot$

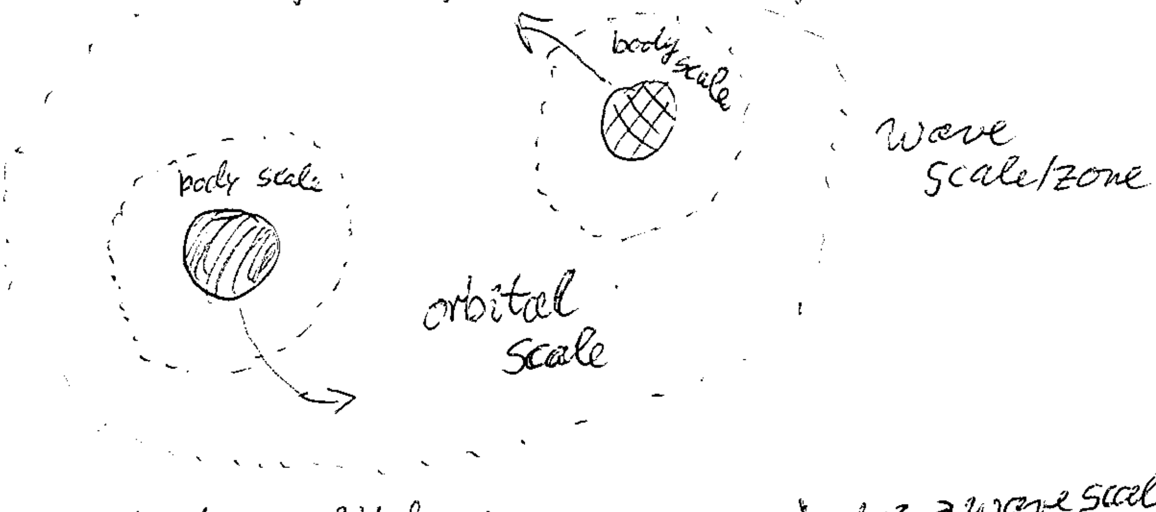
frequency goes up
& amplitude "
$\rightsquigarrow$ like <u>chirp</u> of birds

until now: Newtonian dynamics $\rightsquigarrow$ only approximate
corrections: post-Newtonian (PN) expansion

$$\underset{\substack{\uparrow \\ \text{3rd Kepler}}}{\frac{v^2}{c^2}} \sim \frac{GM}{c^2 r} \sim \epsilon_{PN} \sim \text{PN expansion parameter}$$

$\sim$ weak field & slow motion approximation

one can work out corrections to: $h_{+\times}$, $\mathcal{L}$, equations of motion / E
not easy: many scales involved!



body $\leftrightarrow$ orbital scale:
- multipoles of body
- tidal deformation
- tidal heating / horizon absorption
- resonances
...

body $\leftrightarrow$ wave scale:
- GW emission
- GW scattered back to orbit: tail effect!
- memory effect