Exercises – Gravitational Waves

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Solutions will be provided at: `http://jan-steinhoff.de/lectures/jess2019/`

1 Quadrupole Formula Estimates

The gravitational wave luminosity is approximately given by the celebrated quadrupole formula

$$\mathcal{L} = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle. \tag{1}$$

A crude dimensional analysis leads to the estimate

$$\mathcal{L} \sim \frac{G}{5c^5} \times \frac{(\text{part of the mass that moves})^2 \times (\text{size of the system})^4}{(\text{time for mass to move through the system})^6}, \tag{2}$$

where we have restored the factors of G and c. It holds

$$\frac{G}{c^5} \approx 10^{-53}\,\mathrm{W}^{-1} \tag{3}$$

(One Watt is $\mathrm{W} = \mathrm{kg}\,\mathrm{m}^2/\mathrm{s}^3$.)

Estimate the gravitational-wave luminosity of:

1. Someone waving his fist. How long does it take to produce a single graviton?
2. A 5×10^5 kg steel rod of 20 m length rotating at 5 Hz. (It's close to breaking apart!)
3. The Earth moving around the Sun. Compare this power to things in everyday life.
4. Two black holes, of $30M_\odot$ each, close to their innermost stable orbit (separation $\sim 6\times$total mass). Give this result in units of $M_\odot c^2/\mathrm{s}$. Estimate the (dimensionless) strain h at a distance R of one astronomical unit and 10 million lightyears (size of our local group of galaxies) using $h \sim G/c^4 \times 2\ddot{Q}/R$.

2 Geodesic Deviation

We consider a free falling observer in the field of a monochromatic plane gravitational wave in the linear approximation. We adopt coordinates spanning the local "laboratory" frame of the observer, i.e., the coordinates (approximately) coincide with times and length measured by the free falling observer. This means that close to the observer $g_{\mu\nu} \approx \eta_{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu} \approx 0$, but the curvature tensor can in general not be neglected. (Which physical principle is linked to the existence of these coordinates?)

Show that the equation of geodesic deviation

$$\frac{D^2 r^\mu}{D\tau^2} = R^\mu{}_{\alpha\beta\nu} u^\alpha u^\beta r^\nu, \tag{4}$$

simplifies to

$$\frac{d^2 r_i}{dt^2} \approx \frac{1}{2} r^j \frac{\partial^2 h_{ji}^{\mathrm{TT}}}{\partial t^2}, \tag{5}$$

for two geodesics initially at rest in the local laboratory system, where r^μ is their separation vector. Make use of the fact that the Riemann tensor $R^\mu{}_{\alpha\beta\nu}$ is invariant under coordinate transformation in the linear approximation [see eq. (14) below] and can be evaluated in TT-gauge. Hint: It holds $r^0 = 0$, $h_{0\mu}^{\mathrm{TT}} = 0$, and eq. (9) below.

Interpret the right hand side of eq. (5) as a field of force. Roughly sketch the field lines for both plus and cross polarizations. Hint: It could be helpful to compute the divergence of the force field. At this point, it makes sense to review the discussion of the ring of test-masses given in the lecture.

Finally, evaluate eq. (4) in the transverse-traceless gauge/coordinate system. What is going on here? Hints: It holds $(u^\mu) = (1, 0, 0, 0) + O(h)$ and $dr^i/d\tau = O(h)$. Why?

Bonus Exercises

3 Linear Approximation

In the linear approximation, the metric is written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with the flat Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. It is assumed that the components of $h_{\mu\nu}$ are small and that $O(h^2)$ terms can be neglected. Indices can be pulled using $\eta_{\mu\nu}$.

3.1 Wave Equation

The Christoffel symbols and the curvature tensor are defined as

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}), \tag{6}$$

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\rho}{}_{\nu\beta}\Gamma^{\mu}{}_{\rho\alpha} - \Gamma^{\rho}{}_{\nu\alpha}\Gamma^{\mu}{}_{\rho\beta}. \tag{7}$$

Show that

$$2\Gamma^{\mu}{}_{\alpha\beta} = \eta^{\mu\nu} (\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}) + O(h^2), \tag{8}$$

$$2R_{\mu\nu\alpha\beta} = \partial_\nu \partial_\alpha h_{\mu\beta} + \partial_\mu \partial_\beta h_{\nu\alpha} - \partial_\mu \partial_\alpha h_{\nu\beta} - \partial_\nu \partial_\beta h_{\mu\alpha} + O(h^2). \tag{9}$$

Show that the Einstein equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$ in the linear approximation read

$$\Box \bar{h}_{\mu\nu} - \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha} - \partial^\alpha \partial_\nu \bar{h}_{\mu\alpha} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} + O(h^2) = -16\pi T_{\mu\nu}. \tag{10}$$

Here we use $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\alpha}{}_{\alpha}$. Note that $\bar{h}^{\alpha}{}_{\alpha} = -h^{\alpha}{}_{\alpha}$, i.e., the "bar operation" flips the sign of the trace (it is a trace-reversal operation). Hence $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{\alpha}{}_{\alpha}$.

3.2 Gauge Transformations

In general relativity, a gauge transformation between coordinates x^μ and x'^μ changes a tensor like the metric $g_{\mu\nu}$ as

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}. \tag{11}$$

For a small gauge transformation $x^\mu = x'^\mu + \xi^\mu(x')$ with $\xi^\mu = O(h)$, confirm that

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + O(h^2), \tag{12}$$

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\alpha \xi^\alpha + O(h^2). \tag{13}$$

(Does it matter if derivatives are taken with respect to x or x' here?)

Show that the Riemann tensor (9) is invariant under such a transformation, i.e.

$$R'_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + O(h^2). \tag{14}$$

Finally, start from an $\bar{h}_{\mu\nu}$ in some gauge, where $f_\mu := \partial^\nu \bar{h}_{\mu\nu}$ is a generic function, and argue that one can find a ξ_μ such that $\partial^\nu \bar{h}'_{\mu\nu} = 0$. This is the harmonic gauge introduced in the lectures.

4 Transverse-Traceless Gauge

1. We consider the homogeneous wave equation in the linear approximation and in harmonic gauge
$$\Box\bar{h}_{\mu\nu} = 0, \quad \partial^\nu\bar{h}_{\mu\nu} = 0. \tag{15}$$
Check that monochromatic plane waves are a solution,
$$\bar{h}_{\mu\nu} = a_{\mu\nu}e^{ik_\alpha x^\alpha} + (\text{c.c.}), \tag{16}$$
with the amplitude $a_{\mu\nu} = a_{\nu\mu} = \text{const}$ and the frequency $\omega = k^0 = -k_0$, if the dispersion relation $k_\alpha k^\alpha = 0$ holds (or $\omega^2 = k_i k^i$) and if $k^\nu a_{\mu\nu} = 0$.

2. A residual gauge freedom is left for plane waves. Show that if $\Box\xi_\mu = 0$, then the gauge-transformed field [cf. eq. (13)]
$$h^{\text{TT}}_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\alpha\xi^\alpha \tag{17}$$
still fulfills $\Box h^{\text{TT}}_{\mu\nu} = 0$ and $\partial^\nu h^{\text{TT}}_{\mu\nu} = 0$. We can then write
$$h^{\text{TT}}_{\mu\nu} = a^{\text{TT}}_{\mu\nu}e^{ik_\alpha x^\alpha} + (\text{c.c.}), \quad \xi_\mu = \frac{1}{i\omega}b_\mu e^{ik_\alpha x^\alpha} + (\text{c.c.}). \tag{18}$$
Confirm that it holds
$$a^{\text{TT}}_{\mu\nu} = a_{\mu\nu} + n_\mu b_\nu + n_\nu b_\mu - \eta_{\mu\nu}n_\alpha b^\alpha, \tag{19}$$
$$n^\nu a^{\text{TT}}_{\mu\nu} = 0. \tag{20}$$
where we defined $n_\mu = k_\mu/\omega$. Check that $n^i n_i = 1$ and $n_0 = -1$.

3. Let us try to impose the conditions $a^{\text{TT}\mu}_{\mu} = 0 = a^{\text{TT}}_{0i}$. Argue that these conditions can be met for some choice of b_μ. (You do not need to solve for b_μ explicitly.) Then show that also $a^{\text{TT}}_{00} = 0 = a^{\text{TT}}_{ij}n^j$. This is the transverse-traceless (TT) gauge,
$$h^{\text{TT}\mu}_{\mu} = 0, \quad h^{\text{TT}}_{0i} = 0, \quad h^{\text{TT}}_{00} = 0, \quad h^{\text{TT}}_{ij}n^j = 0. \tag{21}$$

4. Argue that the relation between the nonzero components of $h^{\text{TT}}_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ must be
$$h^{\text{TT}}_{ij} = \Lambda_{ij}{}^{kl}\bar{h}_{kl}, \quad \Lambda_{ij}{}^{kl} = \frac{1}{2}(\Lambda_i{}^k\Lambda_j{}^l + \Lambda_i{}^l\Lambda_j{}^k - \Lambda_{ij}\Lambda^{kl}), \tag{22}$$
where $\Lambda_{ik} = \delta_{ik} - n_i n_k$. Here we have encountered the TT-projector $\Lambda_{ij}{}^{kl}$.

   Alternatively, but lengthy **(optional)**: Explicitly solve for b_μ and show that eq. (19) leads to eq. (22). Hint: From $n^\nu a_{\mu\nu} = 0$ it follows that $a_{\mu 0} = -a_{\mu j}n^j$ and $a^\mu{}_\mu = \Lambda^{ij}a_{ij}$. You should find
$$b_0 = -\frac{1}{2}a_{0j}n^j + \frac{1}{4}a^\mu{}_\mu \qquad = \frac{1}{2}a_{kl}n^k n^l + \frac{1}{4}\Lambda^{kl}a_{kl}, \tag{23}$$
$$b_i = a_{0i} - \frac{1}{2}a_{0j}n^j n_i + \frac{1}{4}n_i a^\mu{}_\mu \qquad = -a_{ij}n^j + \frac{1}{2}a_{kj}n^k n^j n_i + \frac{1}{4}n_i\Lambda^{kl}a_{kl}. \tag{24}$$

Note: Most of the above is only valid for nonstatic solutions $\omega \neq 0$ (“proper waves”).