JESS 2019 L1P1 jan-steinhoff.de/lectures/jess20

Intro: $\mu, \nu, \ldots = 0,1,2,3$ } indices
$i, j, \ldots = 1,2,2$

$\eta = \mathrm{diag}(-1,1,1,1)$ $G = 1 = c$

Gravitational waves (GWs)

max. speed: c
↳ no instantaneous forces
↳ propagation, waves !
↳ electromag., gravitational

particle physics
↳ relativity + quantum mechanics
↳ forces ⇝ bosonic particles ⟷ waves (duality)
graviton: massless spin2 particle
↳ 2 polarizations!

linearized gravity

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$
↑ flat spacetime

Indices now pulled with η !

in $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$ (exercise 3.1)

$$\rightsquigarrow \underbrace{\partial_\alpha \partial^\alpha}_{\Box} \bar{h}_{\mu\nu} - \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha} - \partial^\alpha \partial_\nu \bar{h}_{\mu\alpha} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\alpha{}_\alpha$ "trace-reversal"

gauge/coordinate fixing: $\partial^\alpha \bar{h}_{\mu\alpha} = 0$

$$\rightsquigarrow \boxed{\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}}$$

vacuum $T_{\mu\nu}=0$ $\rightsquigarrow$ plane-wave solutions

transverse-traceless (TT)-gauge for monochromatic plane waves:

e.g. wave in z-direction (exercise 4)

$$(h_{\mu\nu}^{TT}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_+ & a_\times & 0 \\ 0 & a_\times & -a_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{a_{ij}^{TT}} e^{ik_\mu x^\mu} + c.c. \qquad (*)$$

k_μ: wave vector $= (-\omega, \vec{k})$

makes polarizations $+,\times$ manifest (physical DOF)

in general: $h_{0\mu}^{TT}=0$, $\partial^j h_{ij}^{TT}=0$, $Tr(h^{TT})=0$ $\rightsquigarrow$ traceless

$k^j a_{ij}^{TT}=0$ $\rightsquigarrow$ transverse

Are GWs observable?

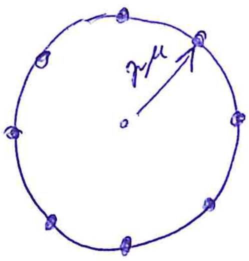
↳ might be a gauge effect

↳ should carry energy, but:

grav. energy can not be localised due to equivalence principle

(also: nonlinearities, are there exact GW sol

look at free-falling ring of test-masses:



geodesic deviation

$$\frac{D^2 r^\mu}{D\tau^2} = R^\mu{}_{\alpha\beta\nu} u^\alpha u^\beta r^\nu$$

u^μ: 4-velocity of ring

τ: proper time " "

t: coordinate time

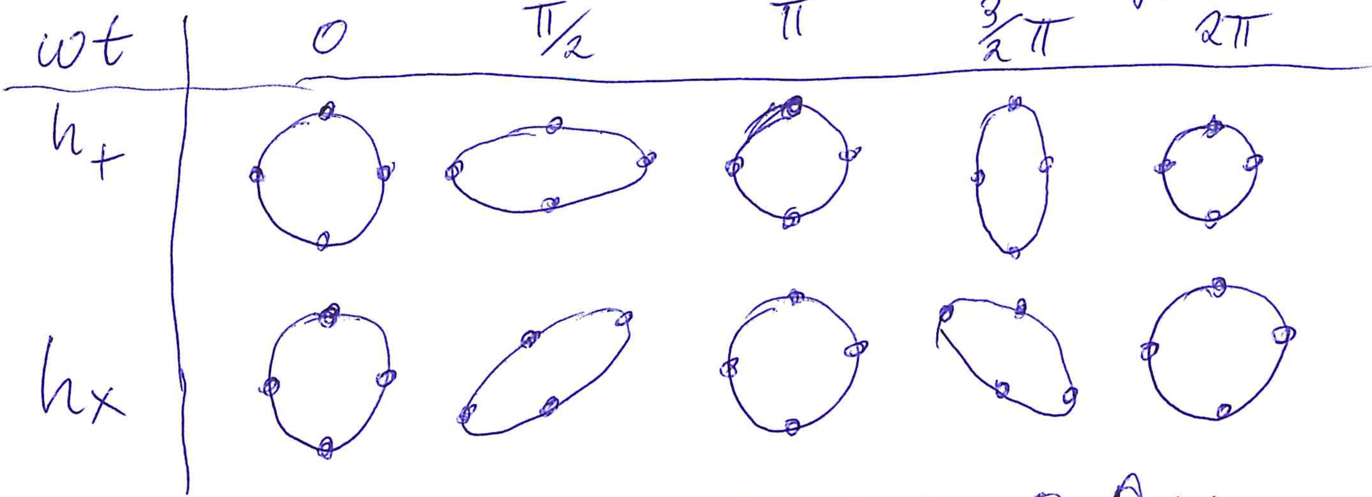
after "some" calculation in linear grav.: (exercise 2)

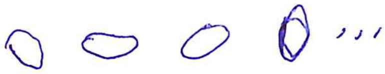
$$\frac{d^2 r_i}{dt^2} \approx \frac{1}{2} r^j \frac{\partial^2 h_{ij}^{TT}}{\partial t^2}$$

ansatz: $r^i = r_0^i + \Delta r^i$, $|\Delta\vec{r}| \ll |\vec{r}_0|$ and (*)

$$\rightsquigarrow \Delta r^i \approx \frac{1}{2} r_0^j h_{ij}^{TT} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_\times & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{r}_0$$

& $h_{+,\times} \sim \sin\omega t$



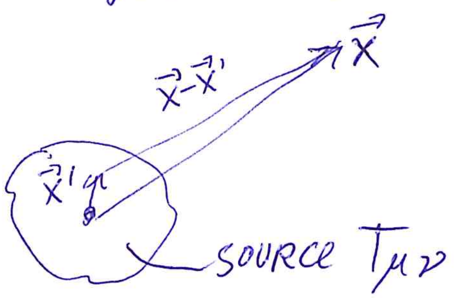
circular pol. from superposition: 

GW detectable ✓

$h \sim$ (dim.len) strain

gravity (linear)	electrodynamics
harmonic gauge $\partial^\alpha \bar{h}_{\mu\alpha} = 0$	Lorenz gauge $\partial^\alpha A_\alpha = 0$
Einstein eq: $\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$	Maxwell eq: $\Box A^\mu = -4\pi j^\mu$
solution $\bar{h}_{\mu\nu} = 4\int d^3x' \frac{T_{\mu\nu}(\vec{x}', t_{ret})}{\lvert\vec{x}-\vec{x}'\rvert}$	$A^\mu = \int d^3x' \frac{j^\mu(\vec{x}', t_{ret})}{\lvert\vec{x}-\vec{x}'\rvert}$
$t_{ret} = t - \lvert\vec{x}-\vec{x}'\rvert$	
quadrupolar radiation	dipolar rad.
2 polarization	2 polarization
⇔ transverse-travellen waves	⇔ transverse waves

far-zone approximation: $|\vec{x}| \gg |\vec{x}'|$



$$\Downarrow \ |\vec{x}-\vec{x}'| \approx |\vec{x}| =: R, \quad t_{ret} \approx t - R$$

$$\bar{h}_{\mu\nu} \approx \frac{4}{R} \int d^3x' \, T_{\mu\nu}(\vec{x}', t_{ret})$$

... some calculation, similar to edyn ...

$$\Downarrow \ \boxed{\bar{h}^{ij} \approx \frac{2}{R} \ddot{M}^{ij}(t_{ret})}$$

where $M^{ij} = \int d^3x \, T^{00} \, x^i x^j$

$\uparrow$ energy/mass density

cp. quadrupole:

$$Q^{ij} = \int d^3x \, T^{00} \left(x^i x^j - \frac{1}{3} \delta^{ij} x^k x_k \right)$$

$$= M^{ij} - \frac{1}{3} M^k{}_k \delta^{ij}$$

project to TT-part: (exercise 4)

$$\boxed{h^{TT}_{ij} \approx \frac{2}{R} \underbrace{\Lambda_{ijkl}}_{\text{TT-projector}} \ddot{Q}^{kl}(t_{ret})}$$

quadruple approximation