HU WS2016 V5S1

Wirkungsprinzip der Gravitationswellen

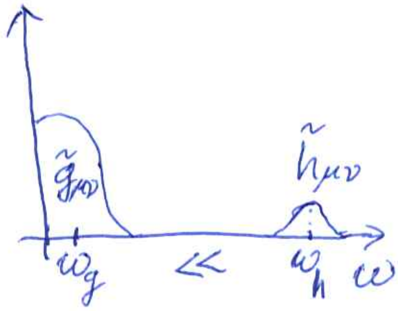
Wiederholung: Wirkung entwickeln in $\delta g_{\mu\nu} = h_{\mu\nu}$

$$S \to S + \delta S + \tfrac{1}{2}\delta^2 S + \mathcal{O}(h^3)$$

$$S = \frac{1}{16\pi}\int d^4x\, \sqrt{-g}\, R \;, \quad \delta S = \frac{1}{16\pi}\int d^4x\, \sqrt{-g}\, G^{\mu\nu}\delta g_{\mu\nu}$$

$$\frac{1}{2}\delta^2 S = -\frac{1}{32\pi}\int d^4x\, \delta\left(\sqrt{-g}\, G^{\mu\nu}\right)\delta g_{\mu\nu} \to \text{heute ausrechnen}$$

nach Variation: - $\delta g_{\mu\nu} = h_{\mu\nu}$ einsetzen
- vereinfachen durch
Separation der Skalen →



$$\mathcal{O}(h):\ \delta S \overset{\delta g = h}{=} -\frac{1}{16\pi}\int d^4x\, \sqrt{-g}\, G^{\mu\nu} h_{\mu\nu}$$

$\hookrightarrow P^{\mu\nu\alpha\beta} R_{\alpha\beta}$

$\hookrightarrow \partial\Gamma + \Gamma\Gamma \sim \partial^2 g + (\partial g)^2$

$\boxed{\delta S \approx 0}$ da $h_{\mu\nu}$ schnell fluktuiert und $\sqrt{-g}\, G^{\mu\nu} \approx$ const auf der Skala ω_h ⇝ $\boxed{g \text{ und } h \text{ entkoppeln auf } \mathcal{O}(h)\ \checkmark}$

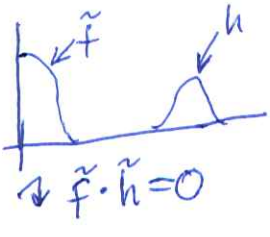
präzise Argumentation durch Fourier-Trafo

$$\int d^4x\, f^{\mu\nu} h_{\mu\nu} = \int \underset{``dt\,d^3x"}{d^4x} \int_{\omega_1}\int_{\omega_2} e^{-it(\omega_1+\omega_2)}\tilde{f}(\omega_1)\tilde{h}(\omega_2)$$

↳ langsam veränderlich
hier: $f^{\mu\nu} = \sqrt{-g}\, G^{\mu\nu}$

$$= \int d^3x \int_{\omega_1}\int_{\omega_2} 2\pi\,\delta(\omega_1+\omega_2)\,\tilde{f}(\omega_1)\tilde{h}(\omega_2)$$

$$= \int d^3x \int_\omega \tilde{f}(\omega)\,\underbrace{\tilde{h}(-\omega)}_{h(\omega),\ \text{da } h\in\mathbb{R}} \approx 0$$



⇝ $\tilde{f}\cdot\tilde{h} = 0$

weitere Annahme für $\mathcal{O}(h^2)$:

$g_{\mu\nu}$ langsam veränderlich in Raum und Zeit

⇝ $\partial g \sim \omega_g\, \mathcal{O}(g)$ ($\mathcal{O}(g) \sim 1$)

$\partial_\alpha h_{\mu\nu} \sim \omega_h \cdot h$ für mindestens eine Wahl der Indizes $\alpha,\mu\nu$

⇝ $\nabla_\alpha h_{\mu\nu} = \underbrace{\partial_\alpha h_{\mu\nu}}_{\sim\omega_h h} + \underbrace{\Gamma h}_{\gg\,\omega_g h} \approx \partial_\alpha h_{\mu\nu}$ ⇝ starke Vereinfachung der Rechnung ✓ Ableitungen von $h_{\mu\nu}$ können vertauscht werden etc.

auch $\underset{\sim\omega_h^2 h}{\partial\partial h} \gg \underset{\omega_g^2 h}{R\, h}$

⇝ Krümmungstensor kann auf $\mathcal{O}(h^2)$ vernachlässigt werden!

→ effektiv ist $g_{\mu\nu} \approx$ const auf $\mathcal{O}(h^2)$

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$\delta R^\mu{}_{\alpha\nu\beta} = \nabla_\nu \delta\Gamma^\mu{}_{\beta\alpha} - \nabla_\beta \delta\Gamma^\mu{}_{\nu\alpha}$

jetzt: $\frac{1}{2}\delta^2 S = -\frac{1}{32\pi}\int d^4x\, \delta(\sqrt{-g}\, G^{\mu\nu})\, \delta g_{\mu\nu}$

$G^{\mu\nu} = P^{\mu\nu\alpha\beta} R_{\alpha\beta} = P^{\mu\nu\alpha\beta} R^\rho{}_{\alpha\rho\beta}$

$= -\frac{1}{32\pi}\int d^4x\, h_{\mu\nu}\left[\delta R^\rho{}_{\alpha\rho\beta}\sqrt{-g}\, P^{\mu\nu\alpha\beta} + R^\rho{}_{\alpha\rho\beta}\, \delta(\sqrt{-g}\, P^{\mu\nu\alpha\beta})\right]$

$\hookrightarrow \partial\delta\Gamma \sim \partial\partial h$ $\quad$ $\hookrightarrow \partial\partial g$ $\quad$ $\hookrightarrow h$

$\omega_h^2 \cdot h \gg \omega_g^2 \cdot h \rightarrow$ vernachlässigbar

$= -\frac{1}{32\pi}\int d^4x\, \sqrt{-g}\, h_{\mu\nu} P^{\mu\nu\alpha\beta}\left(\nabla_\rho \delta\Gamma^\rho{}_{\beta\alpha} - \nabla_\beta \delta\Gamma^\rho{}_{\rho\alpha}\right)$

$\nabla_\rho \to \partial_\rho$, $\nabla_\beta \to \partial_\beta$

part. Int. ~~$= +\frac{1}{32\pi}\int d^4x\sqrt{-g}\, P$~~

NR: $\delta\Gamma^\rho{}_{\beta\alpha} = \frac{1}{2} g^{\rho\sigma}(\partial_\beta h_{\alpha\sigma} + \partial_\alpha h_{\beta\sigma} - \partial_\sigma h_{\alpha\beta}) + \underbrace{h\,\partial g}_{\text{vernachl.}}$

$= \underbrace{g^{\rho\sigma}\partial_{(\beta} h_{\alpha)\sigma}}_{\partial_{(\beta}h_{\alpha)}{}^\rho + h\partial g} - \frac{1}{2}\partial^\rho h_{\alpha\beta}$

$= \partial_{(\beta} h_{\alpha)}{}^\rho - \frac{1}{2}\partial^\rho h_{\alpha\beta}$

$\delta\Gamma^\rho{}_{\rho\alpha} = \frac{1}{2}\partial_\rho h_\alpha{}^\rho + \frac{1}{2}\partial_\alpha h_\rho{}^\rho - \frac{1}{2}\partial^\rho h_{\alpha\rho} = \frac{1}{2}\partial_\alpha h$

$\frac{1}{2}\delta^2 S = -\frac{1}{32\pi}\int d^4x\sqrt{-g}\, h_{\mu\nu} P^{\mu\nu\alpha\beta}\left(-\frac{1}{2}\partial_\rho\partial^\rho h_{\alpha\beta} + \partial_\rho \partial_{(\beta} h_{\alpha)}{}^\rho - \frac{1}{2}\partial_\beta\partial_\alpha h\right)$

sym in α,β $\xrightarrow{()\ \text{weg}}$ $\partial_\rho\partial_\beta\underbrace{(h_\alpha{}^\rho - \frac{1}{2}\delta^\rho_\alpha h)}_{\bar{h}_\alpha{}^\rho}$

$= -\frac{1}{32\pi}\int d^4x\sqrt{-g}\, h_{\mu\nu} P^{\mu\nu\alpha\beta}\left(-\frac{1}{2}\partial_\rho\partial^\rho h_{\alpha\beta} + \partial_\rho\partial_\beta \bar{h}_\alpha{}^\rho\right)$

$\frac{1}{2}\left(-h_{\mu\nu}P^{\mu\nu\alpha\beta}\partial_\rho\partial^\rho h_{\alpha\beta} + h_{\mu\nu}P^{\mu\nu\alpha\beta}\cdot 2\,\partial_\rho\partial_\beta \bar{h}_\alpha{}^\rho\right)$

$P^{\alpha\beta\mu\nu} h_{\mu\nu} = \bar{h}^{\alpha\beta}$

part. Int. $= +\frac{1}{64\pi}\int d^4x\sqrt{-g}\left(-\partial_\rho h_{\mu\nu} P^{\mu\nu\alpha\beta}\partial^\rho h_{\alpha\beta} + 2\,\partial_\beta \bar{h}^{\alpha\beta}\,\partial_\rho \bar{h}_\alpha{}^\rho\right) + h\,\partial(P\sqrt{-g})$

$2\,\partial^\mu \bar{h}_{\mu\nu}\, g^{\nu\beta}\, \partial^\alpha \bar{h}_{\alpha\beta}$

~~→ siehe Ü.blatt 4~~

Bewegungsgleichung für $h_{\mu\nu}$
⇝ Variation von S ⇝ Wellengleichung (Ü.blatt 4)

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Brauchen Eichfixierungsterm in der Wirkung (wie in der Elektrodynamik)

wir wählen: (Begründung folgt)

$$S_{EF} = -\frac{1}{16\pi} \int d^4x \frac{\sqrt{-g}}{2} \partial^\mu \bar{h}_{\mu\nu} g^{\nu\beta} \partial^\alpha \bar{h}_{\alpha\beta}$$

$$\leadsto S + S_{EF} = \frac{1}{64\pi} \int d^4x \sqrt{-g}\, P^{\mu\nu\alpha\beta} \partial_\rho h_{\mu\nu} \partial^\rho h_{\alpha\beta} + \frac{1}{16\pi} \int d^4x \sqrt{-g}\, R$$

$P^{\mu\nu\alpha\beta} = P^{\alpha\beta\mu\nu}$ (Ü. blatt 4) $\Longrightarrow$ symmetrisch

Faktor 2 in Variation

Variation von $h_{\mu\nu}$:

$$\delta(S+S_{EF}) = -\frac{1}{64\pi} \int d^4x \sqrt{-g}\, P^{\mu\nu\alpha\beta}\, 2\, \partial^\rho h_{\alpha\beta} \partial^\rho \delta h_{\mu\nu}$$

$$= +\frac{1}{32\pi} \int d^4x \sqrt{-g} \underbrace{\left(P^{\mu\nu\alpha\beta} \partial_\rho \partial^\rho h_{\alpha\beta}\right)}_{\overset{!}{=}0} \delta h_{\mu\nu}$$

$$\leadsto 0 = P^{\mu\nu\alpha\beta} \partial_\rho \partial^\rho h_{\alpha\beta} = \partial_\rho \partial^\rho \bar{h}^{\mu\nu}$$

Wellengleichung für $\bar{h}^{\mu\nu}$

in der Eichung $\partial_\mu \bar{h}^{\mu\nu} = 0$ → siehe Ü. blatt 4

Die Eichung

Wiederholung: Koordinatentrafo der Metrik $g_{\mu\nu}$

$$g_{\mu\nu} \longrightarrow g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$

infinitesimale Trafo: $x^\mu = x'^\mu + \xi^\mu$ ↳ klein, beliebige Funktion von x'^μ

$$g_{\mu\nu} \longrightarrow g_{\alpha\beta}(x') \left(\delta^\alpha_\mu + \partial^\alpha \xi_\mu\right)\left(\delta^\beta_\nu + \partial^\beta \xi_\nu\right) + \xi^\rho \partial_\rho g_{\mu\nu}$$

jetzt für $h_{\mu\nu}$: ξ^μ sei hochfrequent ⇝ $\partial_\rho g_{\mu\nu}$ vernachlässigbar

$$\delta h_{\mu\nu} = g_{\alpha\nu} \partial^\alpha \xi_\mu + g_{\mu\beta} \partial^\beta \xi_\nu$$

$$= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad \text{Eichtrafo für } h_{\mu\nu}$$

Wirkung S muss per Konstruktion invariant unter dieser Trafo sein (wird explizit auf Blatt 4 geprüft)