EFT calculation of the gravitational binary effective action at 1PN

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Contents

1. Literature

Effective Field Theory (EFT) program to classical gravitational dynamics and radiation was initiated in references [1, 2], for a review of the literature see references [3–5]. The first post-Newtonian (1PN) Lagrangian was first derived in reference [1] using the EFT approach, here we follow reference [6]. A Mathematica code was published in reference [7].

The lecture notes can be found at `http://jan-steinhoff.de/lectures/tgwg2018/`. We use units with $G = 1 = c$ and the signature of spacetime is mostly minus.

2. Review of path integrals

The aim of this section is to recall some basics about path integrals and how they connect to Feynman rules. A more detailed, didactic treatment can be found in many quantum field theory textbooks. However, one could skip this topic/section and directly try to convert the diagrams in the exercise section 5 into integrals using the Feynman rules, which gives one a taste of how the machinery works.

In what follows, we recall the path integral method for a single scalar field φ. The effective action S_{eff} can be obtained by integrating out φ from the full action S, as in

$$\text{const} \cdot e^{\frac{i}{\hbar} S_{\text{eff}}} = \int \mathcal{D}\varphi \, e^{\frac{i}{\hbar} S}. \tag{1}$$

Let us focus on terms in S quadratic in the fields, which lead to the propagators. A term $-\frac{C}{2}\varphi\Delta\varphi$ in the action leads to a propagator normalized as $C^{-1}\Delta^{-1}$ due to the formula for Gaussian integrals applied to the generating functional of the free field Z_0, e.g.

$$Z_0[J] = \int \mathcal{D}\varphi \, e^{\int d^4x \left[-\frac{C}{2}\varphi\Delta\varphi + J\varphi\right]} = Z_0[0] e^{\int d^4x \, \frac{1}{2C} J\Delta^{-1} J}, \tag{2}$$

$$\begin{aligned} \langle \varphi(x_1)\varphi(x_2)\rangle &= \frac{1}{Z_0[0]} \int \mathcal{D}\varphi \, \varphi(x_1)\varphi(x_2) e^{\int d^4x \left[-\frac{C}{2}\varphi\Delta\varphi\right]}, &(3)\\ &= \frac{1}{Z_0[0]} \left. \frac{\delta^2 Z[J]}{\delta J(x_2)\delta J(x_1)} \right|_{J=0}, &(4)\\ &= \frac{1}{C}\Delta^{-1}\delta(x_1 - x_2). &(5) \end{aligned}$$

In QFT it is customary to include a factor $\frac{\hbar}{i}$ in the propagator coming from the exponent $\frac{i}{\hbar}S$ in the path integral, which we drop here since all of these factors cancel in the classical limit.

All terms in the action (except the propagator ones) are represented by vertices. Let us recall how this works for a φ^4 interaction. The path integral then reads

$$\begin{aligned} \int \mathcal{D}\varphi \, e^{\int d^4x \left[-\frac{C}{2}\varphi\Delta\varphi + \frac{\lambda}{4!}\varphi^4\right]} &= \int \mathcal{D}\varphi \, e^{\int d^4x \left[-\frac{C}{2}\varphi\Delta\varphi\right]} \sum_n \frac{1}{n!} \left[\frac{\lambda}{4!}\varphi^4\right]^n, &(6)\\ &= \sum_n \frac{1}{n!} \left[\frac{\lambda}{4!}\frac{\delta^4}{\delta J^4}\right]^n Z_0[J] \bigg|_{J=0}. &(7) \end{aligned}$$

The contributions to this sum are visualized by Feynman diagrams, where $\frac{\lambda}{4!}\varphi^4$ or $\frac{\lambda}{4!}\frac{\delta^4}{\delta J^4}$ is represented by a vertex with 4 lines, and each propagator generated by a pair of derivatives acting on $Z[J]$ is represented by a line connecting the vertices; nonzero contributions must have all lines emanating from the vertices connected due to setting $J = 0$ in the end. The overall constant factor $Z[0]$ is irrelevant. The factors $1/4!$ and $1/n!$ usually cancel for combinatorial reasons, e.g., there are $4!$ different ways to connect

4 distinct lines to a vertex. This is not true when the diagram has symmetries, leading to the notion of the symmetry factor (formally, the size of the automorphism group of the diagram). In the end, the definition of the effective action (1) reads

$$e^{S_{\text{eff}}} = \sum_n \frac{1}{n!} \left[\frac{\lambda}{4!} \frac{\delta^4}{\delta J^4} \right]^n e^{\int d^4x \, \frac{1}{2C} J \Delta^{-1} J} \bigg|_{J=0} . \tag{8}$$

One can directly compute S_{eff} by only summing connected diagrams, i.e. diagrams where all vertices can be reached by walking along the propagator lines. This can be shown noticing that the sum of all diagrams (equal to $e^{S_{\text{eff}}}$) can be written in terms of all connected diagrams $\mathcal{C}(i)$ as $\prod_i e^{\mathcal{C}(i)} = e^{\sum_i \mathcal{C}(i)}$.

3. Feynman rules

Now, the aim is to calculate the classical part of the effective action S_{eff} of a binary to 1PN order using a path integral

$$\text{const} \cdot e^{\frac{i}{\hbar} S_{\text{eff}}} = \int \mathcal{D}g \, e^{\frac{i}{\hbar} S}, \tag{9}$$

where S is the complete action with two point masses, Einstein-Hilbert, and gauge fixing terms. (Ghost fields are only need for quantum corrections.) Since we focus on the conservative part, we can ignore the (dissipative) radiation modes, so the metric only contains orbital scale modes (the body-scale modes were integrated out already at this stage). The path integral is evaluated perturbatively using Feynman diagrams, as explained in the last section. We use the Kaluza-Klein ansatz for the metric [6]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{10}$$

$$= -e^{-2V} (dt + 4V_i dx^i)^2 + e^{2V} (\delta_{ij} + \sigma_{ij}) dx^i dx^j. \tag{11}$$

3.1. Propagators

The leading order quadratic terms in the action

$$S_{(2)} = \frac{1}{8\pi} \int d^4x [V \Delta V - 4 V_i \Delta V_i], \tag{12}$$

lead to the propagators (σ_{ij} is not needed at 1PN)

$$\text{———} = \langle V(\boldsymbol{x}_1, t_1) V(\boldsymbol{x}_2, t_2) \rangle, \tag{13}$$

$$= -4\pi \Delta^{-1} \delta(\boldsymbol{x}_1 - \boldsymbol{x}_2) \delta(t_1 - t_2), \tag{14}$$

$$= 4\pi \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)}}{\boldsymbol{k}^2} \delta(t_1 - t_2), \tag{15}$$

$$\text{-----} = \langle V_i(\boldsymbol{x}_1, t_1) V_j(\boldsymbol{x}_2, t_2) \rangle, \tag{16}$$

$$= -\pi \delta_{ij} \delta(t_1 - t_2) \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)}}{\boldsymbol{k}^2}. \tag{17}$$

Quadratic terms involving times derivatives are treated perturbatively, i.e., as vertices.

3.2. Field Vertices

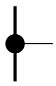
We identify the field vertices directly by the corresponding term in the action

$$= \frac{1}{2!}\frac{1}{4\pi}\int d^4x\,\partial_0 V \partial_0 V, \tag{18}$$

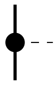
Notice that the usual procedure is to drop the factors $1/i!$ for i identical lines on a vertex and multiply the diagram by 1/(symmetry factor) instead. In QFT, it is customary to present the vertex rules with the $1/i!$ already dropped.

3.3. Worldline vertices

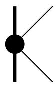
The worldline couplings are to 1PN order



$$= m\int dt\, V(\boldsymbol{x}(t),t)\left[1+\frac{3}{2}\boldsymbol{v}^2\right], \tag{19}$$



$$= -4m\int dt\, V_i(\boldsymbol{x}(t),t)\, v^i, \tag{20}$$



$$= -\frac{1}{2!}m\int dt\,[V(\boldsymbol{x}(t),t)]^2, \tag{21}$$

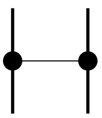
where $\boldsymbol{v} = \dot{\boldsymbol{x}} = d\boldsymbol{x}/dt$. The worldlines are represented by thick lines, but note that there are no propagators associated to them. The field-independent parts of the action

$$m\int dt\left[-1+\frac{1}{2}\boldsymbol{v}^2+\frac{1}{8}\boldsymbol{v}^4\right], \tag{22}$$

can just be taken over to S_{eff} unchanged, one copy for each body.

4. Example: single gravito-electric exchange at 1PN

The Feynman diagrams can be translated into integrals by "pasting" in the right-hand sides of the Feynman rules; fields are contracted using $\langle\ldots\rangle$ according to the propagator lines. For instance, an exchange of a single "electric" graviton V between the worldlines is given by the diagram



$$\begin{aligned}
&= \int dt_1\, m_1\left[1+\frac{3}{2}\boldsymbol{v}_1^2(t_1)\right]\int dt_2\, m_2\left[1+\frac{3}{2}\boldsymbol{v}_2^2(t_2)\right]\langle V(\boldsymbol{x}_1(t_1),t_1)V(\boldsymbol{x}_2(t_2),t_2)\rangle,\\
&\approx m_1 m_2\, 4\pi\int dt_1 dt_2\frac{d^3\boldsymbol{k}}{(2\pi)^3}\left[1+\frac{3}{2}\boldsymbol{v}_1^2(t_1)+\frac{3}{2}\boldsymbol{v}_2^2(t_2)\right]\frac{e^{i\boldsymbol{k}\cdot[\boldsymbol{x}_1(t_1)-\boldsymbol{x}_2(t_2)]}}{\boldsymbol{k}^2}\delta(t_1-t_2),\\
&= \int dt\,\frac{m_1 m_2}{r(t)}\left[1+\frac{3}{2}\boldsymbol{v}_1^2(t)+\frac{3}{2}\boldsymbol{v}_2^2(t)\right],
\end{aligned} \tag{23}$$

where $r = |\boldsymbol{x}_1 - \boldsymbol{x}_2|$. We have dropped terms higher than quadratic in the velocities, since these would be 2PN. Useful integrals can be found in appendix A. Notice that the result is the Newtonian gravitational potential, with some velocity corrections at 1PN.

5. Exercise

To order 1PN in GR, the effective action S_{eff} is the sum of (22) for each body, of (23), and of the following diagrams,

(24)

Calculate these diagrams and obtain the 1PN action. (The last diagram has a symmetry factor of 2.)

A. Integrals

A useful integral is

$$\int \frac{d^d\boldsymbol{k}}{(2\pi)^d} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{(\boldsymbol{k}^2)^\alpha} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)} \left(\frac{\boldsymbol{x}^2}{4}\right)^{\alpha-d/2}, \tag{25}$$

explicitly for e.g. $d=3$, $\alpha=1,2$,

$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{\boldsymbol{k}^2} = \frac{1}{4\pi|\boldsymbol{x}|} = -\Delta^{-1}\delta(\boldsymbol{x}), \tag{26}$$

$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{\boldsymbol{k}^4} = -\frac{|\boldsymbol{x}|}{8\pi}. \tag{27}$$

References

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[5] M. Levi, "Effective Field Theories of Post-Newtonian Gravity: A comprehensive review," arXiv:1807.01699 [hep-th].

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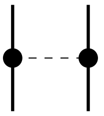
[7] M. Levi and J. Steinhoff, "EFTofPNG: A package for high precision computation with the Effective Field Theory of Post-Newtonian Gravity," *Class. Quant. Grav.* **34** no. 24, (2017) 244001, `arXiv:1705.06309 [gr-qc]`.

B. Solution

The first diagram reads

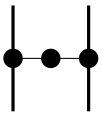


$$= \int dt_1 4m_1 v_1^i(t_1) \int dt_2\, 4m_1 v_2^i(t_2) \langle V_i(\boldsymbol{x}_1(t_1), t_1) V_j(\boldsymbol{x}_2(t_2), t_2)\rangle, \tag{28}$$

$$= -\int dt_1 dt_2 \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, 16\pi m_1 m_2 v_1^i(t_1) v_2^j(t_2) \delta_{ij} \frac{e^{i\boldsymbol{k}\cdot[\boldsymbol{x}_1(t_1)-\boldsymbol{x}_2(t_2)]}}{\boldsymbol{k}^2}, \tag{29}$$

$$= -\int dt\, \frac{4m_1 m_2}{r} \boldsymbol{v}_1 \cdot \boldsymbol{v}_2, \tag{30}$$

where we have suppressed the time dependence in the final result. The next diagram translates into the integral [dropping the factorial prefactor in equation (18)]



$$= \int dt_1 dt_2 d^3\boldsymbol{x} dt\, \frac{m_1 m_2}{4\pi} \langle V(\boldsymbol{x}_1(t_1), t_1) \partial_t V(\boldsymbol{x}, t)\rangle \langle V(\boldsymbol{x}_1(t_1), t_1) \partial_t V(\boldsymbol{x}, t)\rangle, \tag{31}$$

$$= 4\pi \int dt_1 dt_2 dt \frac{d^3\boldsymbol{k}_1}{(2\pi)^3} \frac{d^3\boldsymbol{k}_2}{(2\pi)^3} d^3\boldsymbol{x}\, m_1 m_2 \frac{e^{i\boldsymbol{k}_1\cdot[\boldsymbol{x}_1(t_1)-\boldsymbol{x}]} e^{i\boldsymbol{k}_2\cdot[\boldsymbol{x}_2(t_2)-\boldsymbol{x}]}}{\boldsymbol{k}_1^2 \boldsymbol{k}_2^2}\, \partial_t \delta(t_1 - t)\, \partial_t \delta(t_2 - t),$$

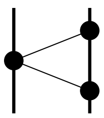
$$= 4\pi \int dt_1 dt_2 dt d^3\boldsymbol{k}_1 \frac{d^3\boldsymbol{k}_2}{(2\pi)^3}\, m_1 m_2 \frac{e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1(t_1)} e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_2(t_2)} \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2)}{\boldsymbol{k}_1^2 \boldsymbol{k}_2^2}\, \partial_{t_1} \delta(t_1 - t)\, \partial_{t_2} \delta(t_2 - t),$$

$$= 4\pi \int dt_1 dt_2 dt \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, m_1 m_2 \frac{e^{i\boldsymbol{k}\cdot[\boldsymbol{x}_1(t_1)-\boldsymbol{x}_2(t_2)]}}{\boldsymbol{k}^4}\, \partial_{t_1} \delta(t_1 - t)\, \partial_{t_2} \delta(t_2 - t), \tag{32}$$

$$= -\int dt_1 dt_2 dt\, \frac{m_1 m_2}{2} \delta(t_1 - t) \delta(t_2 - t) \partial_{t_1} \partial_{t_2} |\boldsymbol{x}_1(t_1) - \boldsymbol{x}_2(t_2)|, \tag{33}$$

$$= \int dt\, \frac{m_1 m_2}{2r} \left(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 - \boldsymbol{v}_1 \cdot \boldsymbol{n}\, \boldsymbol{v}_2 \cdot \boldsymbol{n}\right), \tag{34}$$

where $\boldsymbol{n} = (\boldsymbol{x}_1 - \boldsymbol{x}_2)/r$. The third diagram has a symmetry factor of 2, since an exchange of the propagator lines leads to the same diagram. It therefore picks up a factor of 1/2 (while again dropping factorial prefactors) and reads

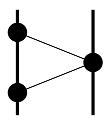


$$= -\frac{1}{2} \int dt_1 dt_2 dt_3\, m_1 m_2^2 \langle V(\boldsymbol{x}_1(t_1), t_1) V(\boldsymbol{x}_2(t_2), t_2)\rangle \langle V(\boldsymbol{x}_1(t_1), t_1) V(\boldsymbol{x}_2(t_3), t_3)\rangle, \tag{35}$$

$$= -\frac{1}{2} \int dt_1 dt_2 dt_3\, m_1 m_2^2\, \frac{\delta(t_1 - t_2)\delta(t_1 - t_3)}{|\boldsymbol{x}_1(t_1) - \boldsymbol{x}_2(t_2)|\, |\boldsymbol{x}_1(t_1) - \boldsymbol{x}_2(t_3)|}, \tag{36}$$

$$= -\int dt\, \frac{m_1^2 m_2}{2r^2}. \tag{37}$$

The last diagram follows from the previous by an exchange of body labels



$$= -\int dt\, \frac{m_1 m_2^2}{2r^2}. \tag{38}$$

The action at 1PN finally reads

$$
\begin{aligned}
S_{\text{eff}} = \int dt \biggl[&- m_1 - m_2 + \frac{m_1}{2}\boldsymbol{v}_1^2 + \frac{m_2}{2}\boldsymbol{v}_2^2 + \frac{1}{8}m_1\boldsymbol{v}_1^4 + \frac{1}{8}m_2\boldsymbol{v}_2^4 \\
&+ \frac{m_1 m_2}{r}\left(1 + \frac{3}{2}\boldsymbol{v}_1^2 + \frac{3}{2}\boldsymbol{v}_2^2 - \frac{7}{2}\boldsymbol{v}_1\cdot\boldsymbol{v}_2 - \frac{1}{2}\boldsymbol{v}_1\cdot\boldsymbol{n}\,\boldsymbol{v}_2\cdot\boldsymbol{n}\right) - \frac{m_1 m_2(m_1+m_2)}{2r^2}\biggr].
\end{aligned}
\tag{39}
$$