Spin and Quadrupole Contributions to the Motion of Astrophysical Binaries

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Outline

Motivation for analytic study of motion in GR

- Gravitational wave experiments: Advanced LIGO in 2014
  (possibly $>$40 detections of binary NS mergers per year)
- Radio astronomy: double pulsar, SKA, ...
  (also optical: WD+WD binary J0651+2844)
- Formation of supermassive BH vs. gravitational recoil ("kick")
- GPB
- Planetary motion

$\Rightarrow$ most gravity experiments require to study the motion!

Possibilities:

- extreme mass ratio approximation, self-force
- Full numeric simulations (still computationally very expensive)
- post-Minkowskian approximation (weak field)
- post-Newtonian (PN) approximation (weak field & slow motion)

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Equations of Motion

$$\frac{Dp_\mu}{d\tau} = 0 - \frac{1}{2} R_{\mu\rho\beta\alpha} u^\rho S^{\beta\alpha} - \frac{1}{6} R_{\nu\rho\beta\alpha;\mu} J^{\nu\rho\beta\alpha}$$

$$\frac{DS^{\mu\nu}}{d\tau} = 2p^{[\mu} u^{\nu]} + \frac{4}{3} R_{\alpha\beta\rho}{}^{[\mu} J^{\nu]\rho\beta\alpha}$$

- Geodesic equation: momentum p_μ
- Mathisson (1937), Papapetrou (1951): spin / dipole $S^{\mu\nu}$
- Dixon ($\sim$1974): quadrupole $J^{\mu\nu\alpha\beta}$, ...
- EOM for p_μ and $S^{\mu\nu}$ follow from theory! $T^{\mu\nu}{}_{;\nu} = 0 \rightsquigarrow$ EOM

Conserved Quantities:

- For a Killing vector field ξ^μ: $E_\xi = p_\mu \xi^\mu + \frac{1}{2} S^{\mu\nu} \nabla_\mu \xi_\nu$
- Neglecting $J^{\mu\nu\alpha\beta}$ etc.: mass $\underline{m} := \sqrt{-p_\mu p^\mu}$ or $m := -u^\mu p_\mu$ (SSC dep.)

  spin-length $S = \sqrt{\frac{1}{2} S_{\mu\nu} S^{\mu\nu}}$

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Gravitational Skeleton

in terms of delta distribution: Tulczyjew (1975); Steinhoff, Puetzfeld (2009)

$$\sqrt{-g}T^{\mu\nu}(x^\sigma) = \int d\tau \Big[u^{(\mu} p^{\nu)} \delta_{(4)} + \left(u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha} + \frac{1}{3} R_{\alpha\beta\rho}{}^{(\mu} J^{\nu)\rho\beta\alpha} \delta_{(4)} - \frac{2}{3} \left(J^{\mu\alpha\beta\nu} \delta_{(4)} \right)_{||(\alpha\beta)} + \ldots \Big]$$

$$u^\mu = \frac{dz^\mu}{d\tau} \qquad \delta_{(4)} = \delta(x^\sigma - z^\sigma)$$

- Point masses only distinguished by a mass $m = \sqrt{-p_\mu p^\mu}$
- Adding a dipole: Spin
- Higher multipoles: Quadrupole, octupole, ... ("finite size effects")
- Dimensional regularization required for self-gravitating objects

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Angular Velocity and Spin

in Newtonian mechanics and special relativity, e.g. Hanson, Regge (1974)

	Newton	special relativity
body-fixed frame	$x^i_{\text{bf}} = \Lambda^{ij} x^j$	
rotational degrees of freedom	$\Lambda^{ki}\Lambda^{kj} = \delta_{ij}$	$\eta_{AB}\Lambda^{A\mu}\Lambda^{B\nu} = \eta^{\mu\nu}$
$\hookrightarrow$ supplementary condition		$\Lambda_{i\mu} p^\mu = 0$
Angular Velocity	$\Omega^{ij} = \Lambda^{ki}\frac{d\Lambda^{kj}}{dt}$	$\Omega^{\mu\nu} = \Lambda_A{}^\mu \frac{d\Lambda^{A\nu}}{d\tau}$
Spin (L: Lagrangian)	$S_{ij} = 2\frac{\partial L}{\partial \Omega^{ij}}$	$S_{\mu\nu} = 2\frac{\partial L}{\partial \Omega^{\mu\nu}}$
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Remark:

- Angular velocity vector is $\Omega^i = \frac{1}{2}\epsilon_{ijk}\Omega^{jk}$. Analogous for spin.

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Spin Action in GR

Westpfahl (1969); Bailey, Israel (1975); Porto (2006); Steinhoff, Schäfer (2009)

- Minimal coupling:

$$\Omega^{\mu\nu} = \Lambda_A{}^{\mu} \frac{D\Lambda^{A\nu}}{d\tau}$$

$$L = m_c \underbrace{\sqrt{-u_\mu u^\mu}}_{u} + \underbrace{\frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu}}_{\rightsquigarrow \frac{1}{2} S^{ij}\partial_i A_j} + \ldots$$

- $m \approx m_c = \text{const}$
- Valid to linear order in spin
- Gravito-magnetic field $A_i \approx -g_{i0}$

Relevance of T^{00}, T^{i0}, T^{ij}

N	mass T^{00}	$\rightsquigarrow$ gravito-electric field
1PN	flow T^{i0}	$\rightsquigarrow$ gravito-magnetic field (A_i)
2PN	stress T^{ij}	$\rightsquigarrow$ 3-dim. tensor field

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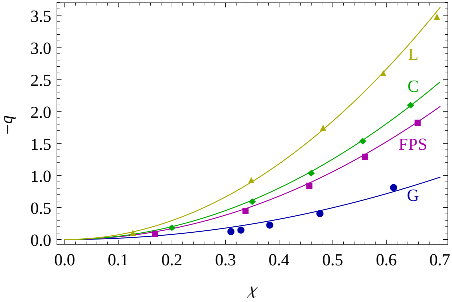
Quadrupole Deformation due to Spin

for neutron stars: Laarakkers, Poisson (1997)

- Here $m = 1.4 M_\odot$
- Dim.-less mass quadrupole: q
- Dim.-less spin: χ
- Quadratic fit is extremely good:

$$-q \approx C_{ES^2} \chi^2$$

- $C_{ES^2} = 4.3 \ldots 7.4$, EOS dependent
- Also depends on mass
- For black holes $C_{ES^2} = 1$



see Laarakkers, Poisson gr-qc/9709033

- higher multipoles: Pappas, Apostolatos (2012)

Tidal Quadrupole Deformation

for NS, e.g. Hinderer & Flanagan (2008); Damour, Nagar (2009); Binnington, Poisson (2009)

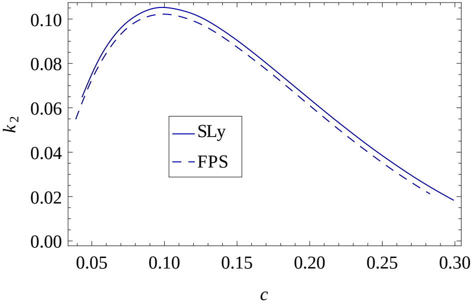
- Linear NS perturbation, thus:

$$-Q = \mu_2 E$$

- Tidal force E (curvature)
- Dim.-less 2nd Love number k_2:

$$k_2 = \frac{3}{2}\frac{\mu_2}{R^5}$$

- Compactness $c = \frac{Gm}{R}$



see Damour, Nagar arXiv:0906.0096

- For certain realistic EOS it holds $k_2 \approx 0.17 - 0.52c$
- For black holes $k_2 = 0$

Quadrupole Action

see e.g. Porto, Rothstein (2008); Goldberger, Rothstein (2006)

$$L_{\text{quad}} = \underbrace{\frac{1}{m_c u} B_{\mu\nu} S^{\mu} u_{\alpha} S^{\alpha\nu}}_{\text{SSC preserving}} + \underbrace{\frac{C_{ES^2}}{2 m_c u} E_{\mu\nu} S^{\mu}{}_{\alpha} S^{\alpha\nu}}_{\text{deformation due to spin}} + \underbrace{\frac{\mu_2}{4u^3} E_{\mu\nu} E^{\mu\nu}}_{\text{tidal deformation}} + \ldots$$

$$E_{\mu\nu} \sim R_{\mu\alpha\nu\beta} u^{\alpha} u^{\beta} \qquad B_{\mu\nu} \sim \tfrac{1}{2} \epsilon_{\mu\rho\alpha\beta} R_{\nu\sigma}{}^{\alpha\beta} u^{\rho} u^{\sigma} \qquad S^{\mu} = \tfrac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_{\nu} S_{\alpha\beta}$$

- m_c, C_{ES^2}, and μ_2: constants, matched to single object
- Now: $m_c \neq m$
- From Bailey, Israel (1975):

$$J^{\mu\nu\alpha\beta} = -6 \frac{\partial L}{\partial R_{\mu\nu\alpha\beta}}$$

- Covariant mass quadrupole: (for $u = 1$)

$$\text{mass quadrupole} \sim 2 \frac{\partial L}{\partial E_{\mu\nu}} = \frac{C_{ES^2}}{m_c} S^{\mu}{}_{\alpha} S^{\alpha\nu} + \mu_2 E^{\mu\nu}$$

Outline

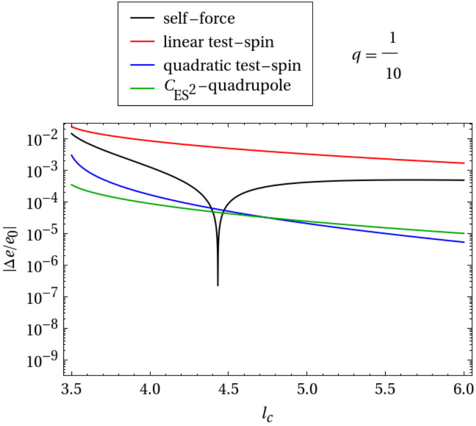
Motion in Schwarzschild background

Steinhoff, Puetzfeld (2012); similar model: Bini, Geralico (2013)

- Conserved quantities:
  E_{∂_t}, E_{∂_ϕ}, S, μ
- Circular orbits, aligned spin
- SSC: $S^{\mu\nu} p_\nu = 0$

$\Rightarrow p_\nu$, $S^{\mu\nu}$ fixed algebratically!

- Binding energy:
  $e(l_c) = E_{\partial_t}/\mu - 1$
- Orbital angular momentum: l_c



spin effects for $\hat{a}_2 = 1$, $C_{ES^2} = 1$

- Taylor-expansion: $e(l_c) = e_0(l_c) + e_1(l_c) + e_2(l_c) + \ldots$
- Scaling: $e_1 \propto q\hat{a}_2$, $e_2^{S^2} \propto -q^2\hat{a}_2^2$, $e_2^{C_{ES^2}} \propto -C_{ES^2} q^2 \hat{a}_2^2$
- Spin-induced quadrupole effects scale like second-order self-force ($\sim q^2$)

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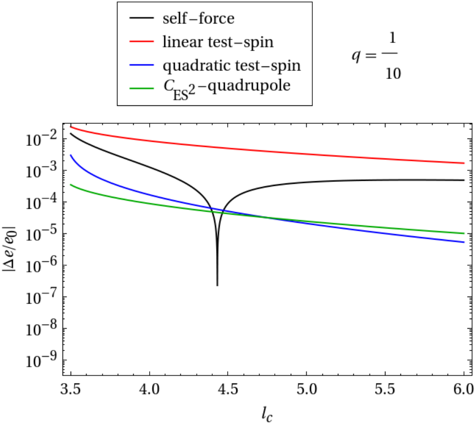

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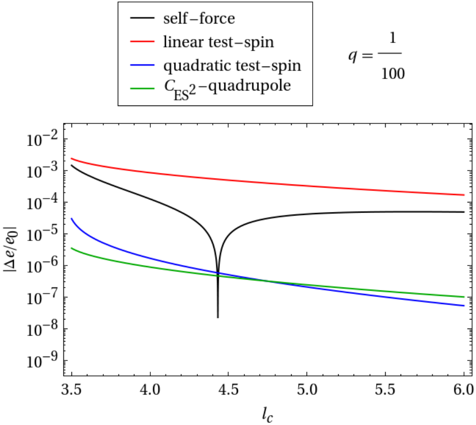

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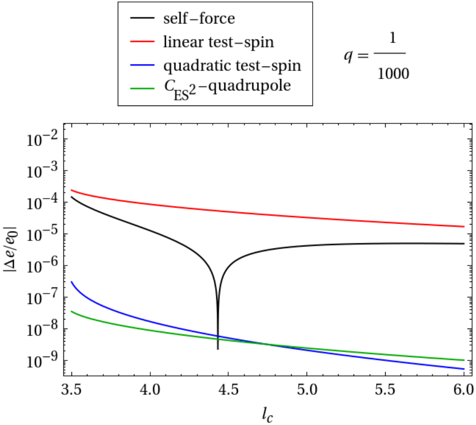

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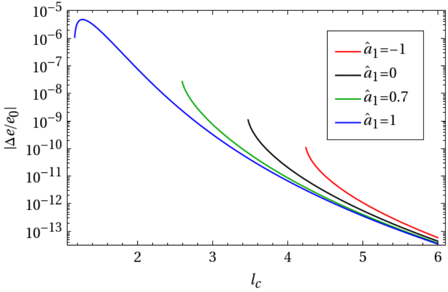
Results for Kerr background

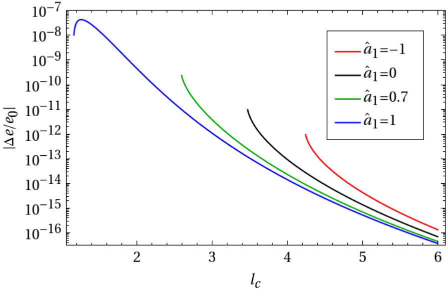
Steinhoff, Puetzfeld (2012)

tidal effects for neutron stars and mass ratio $q = \frac{1}{50}$

$(k_2 = 0.1,\, j_2 = -0.01,\, \hat{R} = 5)$



gravito-electric tidal effects



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- Scaling: $e_2^{k_2} \propto -k_2 q^4 \hat{R}^5$ $\qquad e_2^{j_2} \propto j_2 q^4 \hat{R}^5$
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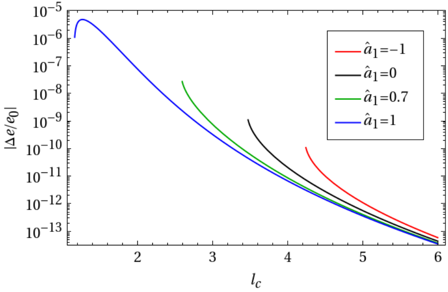
Results for Kerr background

Steinhoff, Puetzfeld (2012)

tidal effects for neutron stars and mass ratio $q = \frac{1}{50}$

$(k_2 = 0.1,\ j_2 = -0.01,\ \hat{R} = 5)$



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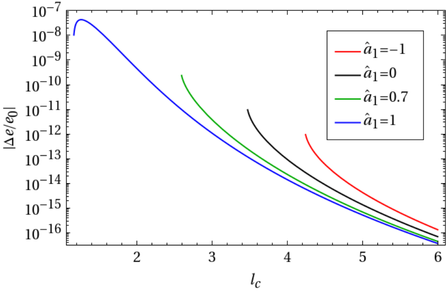

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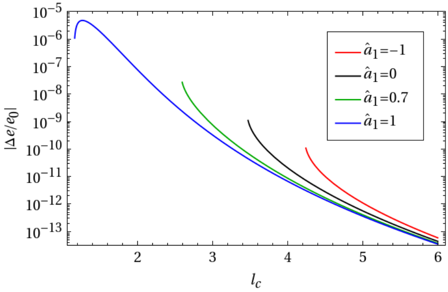

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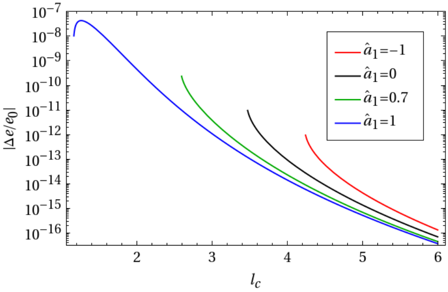

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Outline

Surface Terms and ADM Hamiltonian

ADM ≙ Arnowitt, Deser, Misner (1960)

- Einstein–Hilbert action plus (Regge–Teitelboim–)York–Gibbons–Hawking ("Trace K") surface term:

$$S_{\text{field}} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}R - \frac{1}{16\pi G}\oint d^3y\,2\sqrt{\gamma}K$$

- ADM energy given by surface integral

$$E_{\text{ADM}} = \frac{1}{16\pi G}\oint d^2s_i[g_{ij,j} - g_{jj,i}]$$

- H^{ADM} ≙ ADM energy E_{ADM} expressed in terms of canonical variables
- Canonical field variables: h_{ij}^{TT}, $\pi^{ij\text{TT}}$ TT ≙ transverse-traceless

DeWitt (1967)

"General relativity is unique among field theories in that its energy may always be expressed as a surface integral."

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Canonical Variables to Linear Order in Spin

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- Method: transform action into the form $\int dt(\dot{q}\,p - H)$
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- Canonical $\hat{z}^i$, $\hat{S}_{ij}$, and $\hat{\Lambda}^{ij}$ are "simple" generalizations of flat space case
- Canonical matter momentum $\hat{p}_i$:

$$p_i = \hat{p}_i + \frac{1}{2}\hat{S}_{kj}\Gamma^{kj}{}_i + \dots$$

cf. electrodynamics: $p_i = \hat{p}_i - qA_i$

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Results so far

from various authors with different methods

for maximally rotating objects: $S = \frac{Gm^2\chi}{c}$ $\chi = 1$

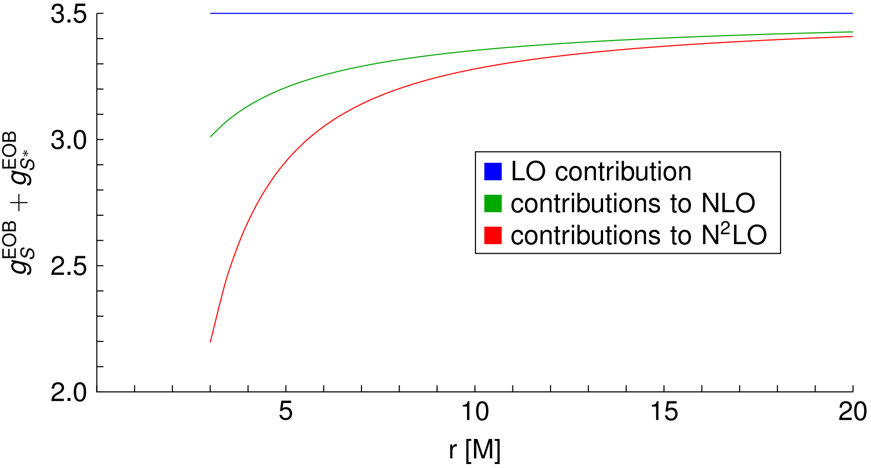
order	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
H^{N}								
PM	$+ H^{1\mathrm{PN}}$		$+ H^{2\mathrm{PN}}$	$+ H^{2.5\mathrm{PN}}$	$+ H^{3\mathrm{PN}}$	$+ H^{3.5\mathrm{PN}}$	$+ \color{red}{H^{4\mathrm{PN}}}$	$+ \color{green}{H^{4.5\mathrm{PN}}}$
SO		$+ H^{\mathrm{LO}}_{\mathrm{SO}}$		$+ H^{\mathrm{NLO}}_{\mathrm{SO}}$		$+ H^{\mathrm{N^2LO}}_{\mathrm{SO}}$	$+ H^{\mathrm{LO,R}}_{\mathrm{SO}}$	$+ \color{red}{H^{\mathrm{N^3LO}}_{\mathrm{SO}}}$
S_1^2			$+ H^{\mathrm{LO}}_{\mathrm{S}_1^2}$		$+ H^{\mathrm{NLO}}_{\mathrm{S}_1^2}$		$+ \color{red}{H^{\mathrm{N^2LO}}_{\mathrm{S}_1^2}}$	$+ \color{green}{H^{\mathrm{LO,R}}_{\mathrm{S}_1^2}}$
$\mathrm{S}_1\mathrm{S}_2$			$+ H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{S}_2}$		$+ H^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{S}_2}$		$+ H^{\mathrm{N^2LO}}_{\mathrm{S}_1\mathrm{S}_2}$	$+ H^{\mathrm{LO,R}}_{\mathrm{S}_1\mathrm{S}_2}$
spin^3						$+ \color{blue}{H^{\mathrm{LO}}_{S^3}}$		$+ \color{red}{H^{\mathrm{NLO}}_{S^3}}$
spin^4							$+ \color{blue}{H^{\mathrm{LO}}_{S^4}}$	
$\vdots$								$\ddots$

H known $\color{green}{\text{EOM known}}$ $\color{blue}{\text{for Black Holes}}$ $\color{red}{\text{not known (yet)}}$

Radiation field known to 2PN order, multipoles to 2.5PN order.

Spin-Orbit: Gyro-Gravitomagnetic Ratios $g_S^{\text{EOB}} + g_{S^*}^{\text{EOB}}$

for equal masses and circular orbits, Nagar (2011); Barausse, Buonanno (2011)



Outline

Newtonian case revised

in preparation, with S. Chakrabarti and T. Delsate

Motivation:

- Adiabatic tidal effects may not be sufficient [Maselli et.al. (2012)]
- Resonances of orbital motion and oscillation modes of the object
- Better understanding of tidal interactions

$$L = \frac{1}{2} m \dot{\mathbf{R}}_*^2 - m \Phi(\mathbf{R}_*) - \frac{1}{2} Q^{ij} \partial_i \partial_j \Phi(\mathbf{R}_*) + \dots$$

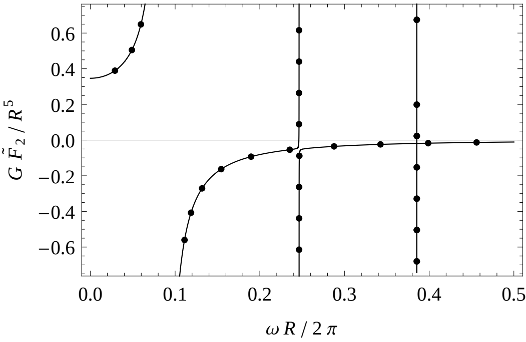
- Idea: response function for Q^{ij}
  [Goldberger, Rothstein, hep-th/0511133]

$$Q^{ij}(t) = -\frac{1}{2} \int dt' \, G^{ij}_{kl}(t, t') \Phi_{,kl}(t')$$

- From Newtonian tidal theory:

$$F(\omega) = \sum_n \frac{I_n^2}{\omega_n^2 - \omega^2}$$

- ω_n are the mode frequencies
- I_n related to overlap integrals



response function $F(\omega)$ for the quadrupole Q^{ij}

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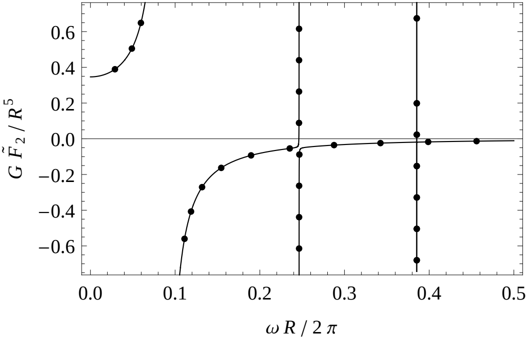

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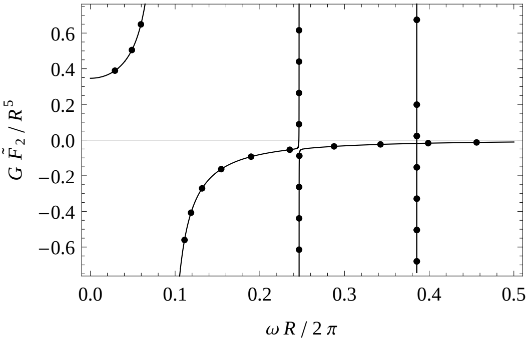

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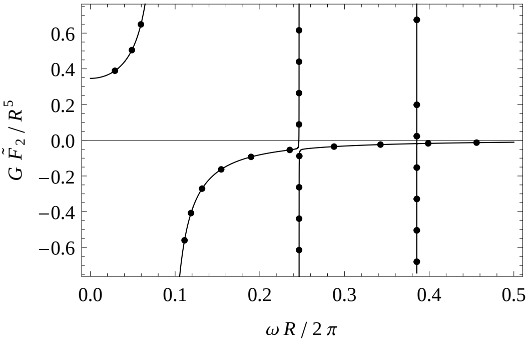

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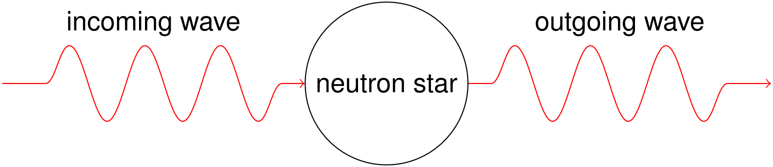
Problems in the relativistic case

- Definition of matter-multipoles in time dependent situations?
  Tentative solution by analogy to optics: need phase shift?



- Quadrupole diverges starting at ω^2, logarithmic scale dependence
  [Goldberger, Ross, arXiv:0912.4254]
- similar: diverging BH Love numbers in higher dimensions
  [Kol, Smolkin, arXiv:1110.3764]
- Dimensional regularization not feasible for NS

Thank you for your attention

and special thanks to my collaborators

Sayan Chakrabarti
Térence Delsate
Johannes Hartung
Steven Hergt
Piotr Jaranowski
Dirk Puetzfeld
Gerhard Schäfer
Manuel Tessmer
Han Wang
Jing Zeng

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