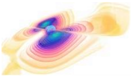
Hamiltonians from the Stress-Energy Tensor

•Jan Steinhoff   Steven Hergt   Gerhard Schäfer

seit 1558

Theoretisch-Physikalisches Institut
Friedrich-Schiller-Universität Jena

DPG Spring Meeting, March 11, 2009



DFG: SFB/TR7 Gravitational Wave Astronomy

Outline

1. Aspects of the ADM Formalism
   - (3+1)-Decomposition
   - ADM Canonical Formalism

2. Hamiltonians from the Stress-Energy Tensor
   - Hamiltonians from Tulczyjew's Stress-Energy Tensor
   - The Stress-Energy Tensor with Quadrupole
   - The Complete NLO S_1^2 Hamiltonian

(3+1)-Decomposition: Metric

- Decomposition of the metric:

$$g_{\mu\nu} = \begin{pmatrix} N^i N_i - N^2 & N_i \\ N_j & \gamma_{ij} \end{pmatrix}, \quad g^{00} = -\frac{1}{N^2}$$

- Normal vector of the 3-dim. hypersurfaces:

$$n_\mu = (-N, 0, 0, 0)$$

- Exterior curvature of the 3-dim. hypersurfaces:

$$K_{ij} \equiv -n_{(i||j)} = -N\Gamma^0_{ij}$$

$$\pi^{ij} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{lj} - \gamma^{ij}\gamma^{kl})K_{kl}$$

(3+1)-Decomposition: Field Equations

- Decomposition of the field equations:
  - Constraint equations:

$$0 = \mathscr{H} \equiv -\frac{1}{16\pi\sqrt{\gamma}}\left[\gamma \mathrm{R} + \frac{1}{2}\left(\gamma_{ij}\pi^{ij}\right)^2 - \gamma_{ij}\gamma_{kl}\pi^{ik}\pi^{jl}\right] + \mathscr{H}^{\text{matter}}$$

$$0 = \mathscr{H}_i \equiv \frac{1}{8\pi}\gamma_{ij}\pi^{jk}{}_{;k} + \mathscr{H}_i^{\text{matter}}$$

  - Evolution equations:

$$\gamma_{ij,0} = 2N\gamma^{-1/2}(\pi_{ij} - \tfrac{1}{2}\gamma_{ij}\gamma_{kl}\pi^{kl}) + N_{i;j} + N_{j;i}$$

$$\begin{aligned}\pi^{ij}{}_{,0} = &-N\sqrt{\gamma}(\mathrm{R}^{ij} - \tfrac{1}{2}\gamma^{ij}\mathrm{R}) + \tfrac{1}{2}N\gamma^{-1/2}\gamma^{ij}(\pi^{mn}\pi_{mn} - \tfrac{1}{2}(\gamma_{mn}\pi^{mn})^2) \\ &-2N\gamma^{-1/2}(\gamma_{mn}\pi^{im}\pi^{nj} - \tfrac{1}{2}\gamma_{mn}\pi^{mn}\pi^{ij}) + \sqrt{\gamma}(N^{;ij} - \gamma^{ij}N^{;m}{}_{;m}) \\ &+(\pi^{ij}N^m)_{;m} - N^i{}_{;m}\pi^{mj} - N^j{}_{;m}\pi^{mi} + \tfrac{1}{2}N\gamma^{im}\gamma^{nj}\sqrt{\gamma}T_{mn}\end{aligned}$$

- Source terms are related to the stress-energy tensor $T^{\mu\nu}$ by:

$$\mathscr{H}^{\text{matter}} \equiv \sqrt{\gamma}T_{\mu\nu}n^\mu n^\nu$$

$$\mathscr{H}_i^{\text{matter}} \equiv -\sqrt{\gamma}T_{i\nu}n^\nu$$

ADM Canonical Formalism

- Gauge independent Hamiltonian:

$$H[x_a^i, p_{ai}, \gamma_{ij}, \pi^{ij}] = \int d^3\mathbf{x}(N\mathscr{H} - N^i\mathscr{H}_i) + E[\gamma_{ij}]$$

$$E[\gamma_{ij}] = \frac{1}{16\pi}\oint d^2 s_i(\gamma_{ij,j} - \gamma_{jj,i})$$

- Hamiltonian in ADMTT gauge (ADM Hamiltonian)
  $\hat{=}$ ADM Energy depending on canonical variables:

$$H_{\mathrm{ADM}} = E[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}, \pi_{\mathrm{TT}}^{ij}] = -\frac{1}{16\pi}\int \mathrm{d}^3\mathbf{x}\,\Delta\phi$$

$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$

- Matter only Hamiltonian: Elimination of h_{ij}^{TT} and π_{TT}^{ij} by solving the evolution equations.

Spin in GR

- Stress-energy tensor density in covariant SSC, $S^{\mu\nu} u_\nu = 0$:

$$\sqrt{-g}\; T^{\mu\nu} = \int d\tau \left[m u^\mu u^\nu \delta_{(4)} - \left(S^{\alpha(\mu} u^{\nu)} \delta_{(4)} \right)_{||\alpha} \right]$$

$$\delta_{(4)} \equiv \delta(x - q(\tau))$$

- EOM follow from $T^{\mu\nu}{}_{||\nu} = 0$:

$$\frac{DS^{\mu\nu}}{d\tau} = 0\,, \quad m\frac{Du_\mu}{d\tau} = \frac{1}{2} S^{\lambda\nu} u^\gamma R^{(4)}_{\mu\gamma\nu\lambda}$$

Source Terms in Canonical Variables

- Calculate matter parts of the constraints:

$$\mathcal{H}^{\text{matter}} = \sqrt{\gamma} T_{\mu\nu} n^\mu n^\nu \,, \qquad \mathcal{H}_i^{\text{matter}} = -\sqrt{\gamma} T_{\nu i} n^\nu$$

- Define canonical momentum p_i as

$$p_i = \int d^3\mathbf{x}\, \mathcal{H}_i^{\text{matter}} \,.$$

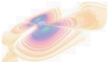
- Define spin $\hat{S}_{ij} = e_{i(k)} e_{j(l)} \varepsilon_{klm} S_{(m)}$ such that $\mathbf{S}^2 = \text{const.}$ and

$$J_{ij} = z^i p_j - z^j p_i + \varepsilon_{ijm} S_{(m)} = \int d^3\mathbf{x} \left(x^i \mathcal{H}_j^{\text{matter}} - x^j \mathcal{H}_i^{\text{matter}} \right)$$

- Go over to canonical position variable $\mathbf{z}$ by a Lie shift (such that one has the Newton-Wigner SSC in flat space).

NLO Spin-Orbit Hamiltonian (DJS 2008)

$$
\begin{aligned}
H_{SO}^{NLO} = & -\frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^2}\left[\frac{5 m_2 \mathbf{p}_1^2}{8 m_1^3} + \frac{3(\mathbf{p}_1 \cdot \mathbf{p}_2)}{4 m_1^2} - \frac{3\mathbf{p}_2^2}{4 m_1 m_2}\right. \\
& \left. + \frac{3(\mathbf{p}_1 \cdot \mathbf{n}_{12})(\mathbf{p}_2 \cdot \mathbf{n}_{12})}{4 m_1^2} + \frac{3(\mathbf{p}_2 \cdot \mathbf{n}_{12})^2}{2 m_1 m_2}\right] \\
& + \frac{((\mathbf{p}_2 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^2}\left[\frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{3(\mathbf{p}_1 \cdot \mathbf{n}_{12})(\mathbf{p}_2 \cdot \mathbf{n}_{12})}{m_1 m_2}\right] \\
& + \frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot \mathbf{p}_2)}{r_{12}^2}\left[\frac{2(\mathbf{p}_2 \cdot \mathbf{n}_{12})}{m_1 m_2} - \frac{3(\mathbf{p}_1 \cdot \mathbf{n}_{12})}{4 m_1^2}\right] \\
& - \frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^3}\left[\frac{11 m_2}{2} + \frac{5 m_2^2}{m_1}\right] \\
& + \frac{((\mathbf{p}_2 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^3}\left[6 m_1 + \frac{15 m_2}{2}\right] + (1 \leftrightarrow 2)
\end{aligned}
$$



NLO Spin$_1$-Spin$_2$ Hamiltonian

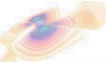
$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_1 S_2} = \frac{1}{2m_1 m_2 r_{12}^3} \Big[& \tfrac{3}{2}((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_2 \times \mathbf{S}_2)\cdot\mathbf{n}_{12}) + \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{p}_2) \\
& + 6((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1 \times \mathbf{S}_2)\cdot\mathbf{n}_{12}) - \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_1) \\
& - 15(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + (\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{p}_2) \\
& - 3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{p}_2) + 3(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) - 3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \Big] \\
+ \frac{3}{2m_1^2 r_{12}^3} \Big[& -((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1 \times \mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
& + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{n}_{12}) \Big] \\
+ \frac{3}{2m_2^2 r_{12}^3} \Big[& -((\mathbf{p}_2 \times \mathbf{S}_2)\cdot\mathbf{n}_{12})((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12}) \\
& + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12}) \Big] \\
+ \frac{6(m_1 + m_2)}{r_{12}^4} \Big[& (\mathbf{S}_1\cdot\mathbf{S}_2) - 2(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12}) \Big]
\end{aligned}
$$

NLO Center of Mass

$$\mathbf{G}^{\mathrm{NLO}}_{\mathrm{SO}} = -\sum_a \frac{\mathbf{p}_a^2}{8m_a^3}(\mathbf{p}_a \times \mathbf{S}_a) + \sum_a \sum_{b\neq a} \frac{m_b}{4m_a r_{ab}} \left[((\mathbf{p}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab}) \frac{5\mathbf{z}_a + \mathbf{z}_b}{r_{ab}} - 5(\mathbf{p}_a \times \mathbf{S}_a) \right] + \sum_a \sum_{b\neq a} \frac{1}{r_{ab}} \left[\frac{3}{2}(\mathbf{p}_b \times \mathbf{S}_a) - \frac{1}{2}(\mathbf{n}_{ab} \times \mathbf{S}_a)(\mathbf{p}_b \cdot \mathbf{n}_{ab}) - ((\mathbf{p}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab}) \frac{\mathbf{z}_a + \mathbf{z}_b}{r_{ab}} \right]$$

$$\mathbf{G}^{\mathrm{NLO}}_{\mathrm{SS}} = \frac{1}{2}\sum_a \sum_{b\neq a} \left\{ [3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b)] \frac{\mathbf{z}_a}{r_{ab}^3} + (\mathbf{S}_b \cdot \mathbf{n}_{ab}) \frac{\mathbf{S}_a}{r_{ab}^2} \right\}$$

$\Rightarrow$ Poincaré algebra is fulfilled.



The Stress-Energy Tensor with Quadrupole

- Stress-energy tensor density with quadrupole has the structure:

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \left[t^{\mu\nu}\delta_{(4)} + (t^{\mu\nu\alpha}\delta_{(4)})_{||\alpha} + (t^{\mu\nu\alpha\beta}\delta_{(4)})_{||\alpha\beta} \right]$$

- Getting expressions for the $t^{\mu\nu\cdots}$ from $T^{\mu\nu}_{\ \ \ ||\nu} = 0$:
  - Dixon's work: Complicated definitions.
  - Tulczyjew's theorems: Complicated calculation.

Ansatz for the Static Source Terms

$$\mathcal{H}^{\text{matter}}_{\mathrm{S}_1^2,\ \text{static}} = \frac{c_1}{m_1}\left(l_1^{ij}\delta_1\right)_{;ij} + \frac{c_2}{m_1}\mathrm{R}_{ij} l_1^{ij}\delta_1 + \frac{c_3}{m_1}\mathbf{S}_1^2\left(\gamma^{ij}\delta_1\right)_{;ij} + \frac{c_4}{m_1}\mathrm{R}\mathbf{S}_1^2\delta_1$$
$$+ \frac{1}{8m_1} g_{mn}\gamma^{pj}\gamma^{ql}\gamma^{mi}{}_{,p}\gamma^{nk}{}_{,q}\hat{S}_{1ij}\hat{S}_{1kl}\delta_1$$
$$+ \frac{1}{4m_1}\left(\gamma^{ij}\gamma^{mn}\gamma^{kl}{}_{,m}\hat{S}_{1ln}\hat{S}_{1jk}\delta_1\right)_{,i}$$

- This ansatz is 3-dim. covariant. Remember:

$$p_i = \int d^3\mathbf{x}\,\mathcal{H}_i^{\text{matter}} = mv_i - \frac{1}{2}g_{ij}\gamma^{lm}\gamma^{jk}{}_{,m}\hat{S}_{kl} + \mathcal{O}(p^2) + \mathcal{O}(\hat{S}^2)$$

- Terms like $l_1^{ij}{}_{;k}\delta_1$ or $l_1^{ij}\delta_{1;k}$ can not appear.
- γ_{ij} for Kerr $\Rightarrow c_1 = -\frac{1}{2}$.
- N for Kerr $\Rightarrow c_2 = 0$.
- c_3 and c_4 do not contribute to the Hamiltonian.

NLO Spin$_1$-Spin$_1$ Hamiltonian

$$
\begin{aligned}
H^{\rm NLO}_{S_1^2} = \frac{1}{r_{12}^3}\Bigg[& \frac{m_2}{4m_1^3}(\mathbf{p}_1\cdot\mathbf{S}_1)^2 + \frac{3m_2}{8m_1^3}(\mathbf{p}_1\cdot\mathbf{n}_{12})^2\mathbf{S}_1^2 - \frac{3m_2}{8m_1^3}\mathbf{p}_1^2(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 \\
& - \frac{3m_2}{4m_1^3}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{S}_1) - \frac{3}{4m_1 m_2}\mathbf{p}_2^2\mathbf{S}_1^2 \\
& + \frac{9}{4m_1 m_2}\mathbf{p}_2^2(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 + \frac{3}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{S}_1^2 \\
& - \frac{9}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 + \frac{3}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})\mathbf{S}_1^2 \\
& - \frac{3}{2m_1^2}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{S}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12}) \\
& + \frac{3}{m_1^2}(\mathbf{p}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{S}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12}) \\
& - \frac{15}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 \Bigg] \\
- \frac{m_2}{r_{12}^4}\Bigg[& \frac{9}{2}(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 - \frac{5}{2}\mathbf{S}_1^2 + \frac{7m_2}{m_1}(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 - \frac{3m_2}{m_1}\mathbf{S}_1^2 \Bigg]
\end{aligned}
$$

Summary

$$H = H_{\mathrm{PM}} + H^{\mathrm{LO}}_{\mathrm{SO}} + H^{\mathrm{LO}}_{\mathrm{S_1S_2}} + H^{\mathrm{LO}}_{\mathrm{S_1^2}} + H^{\mathrm{LO}}_{\mathrm{S_2^2}} + H^{\mathrm{NLO}}_{\mathrm{SO}} + H^{\mathrm{NLO}}_{\mathrm{S_1S_2}} + H^{\mathrm{NLO}}_{\mathrm{S_1^2}} + H^{\mathrm{NLO}}_{\mathrm{S_2^2}} + \dots$$

$H^{\mathrm{NLO}}_{\mathrm{SO}}$: T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **77**, 064032 (2008).

$H^{\mathrm{NLO}}_{\mathrm{S_1S_2}}$: J. Steinhoff, S. Hergt, and G. Schäfer, Phys. Rev. D **77**, 081501(R) (2008);

J. Steinhoff, G. Schäfer, and S. Hergt, Phys. Rev. D **77**, 104018 (2008).

$H^{\mathrm{NLO}}_{\mathrm{S_a^2}}$: S. Hergt and G. Schäfer, Phys. Rev. D **78**, 124004 (2008);

J. Steinhoff, S. Hergt, and G. Schäfer, Phys. Rev. D **78**, 101503(R) (2008).