Gravitational Quadrupole Contributions to the Equations of Motion of Compact Objects

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Motion in General Relativity

Motivation:

- Gravitational wave experiments: Advanced LIGO in 2015
  (possibly $>$40 detections of binary NS mergers per year)
- Pulsar timing via radio astronomy: double pulsar, SKA, ...
  (also optical: WD+WD binary J0651+2844)
- Formation of supermassive BH vs. gravitational recoil ("kick")
- Gravity Probe B
- SgrA*, LRR, Planetary motion, ...

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Possibilities: analytic approximations and/or numeric simulations
Here: small mass ratio approximation

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geodesic motion self-force corrections from internal structure

EOM for test-bodies with internal structure

see Steinhoff, Puetzfeld (2010) for a derivation using Tulczyjew's method

$$\frac{Dp_\mu}{d\tau} = 0 - \frac{1}{2} R_{\mu\rho\beta\alpha} u^\rho S^{\beta\alpha} - \frac{1}{6} R_{\nu\rho\beta\alpha;\mu} J^{\nu\rho\beta\alpha}$$

$$\frac{DS^{\mu\nu}}{d\tau} = 2p^{[\mu} u^{\nu]} + \frac{4}{3} R_{\alpha\beta\rho}{}^{[\mu} J^{\nu]\rho\beta\alpha}$$

$$\frac{DJ^{\mu\nu\alpha\beta}}{d\tau} = ???$$

- Geodesic equation: momentum p_μ
- Mathisson (1937), Papapetrou (1951): spin / dipole $S^{\mu\nu}$
- Dixon (~1974): quadrupole $J^{\mu\nu\alpha\beta}$, ...
- EOM for p_μ and $S^{\mu\nu}$ follow from theory! $T^{\mu\nu}{}_{;\nu} = 0 \rightsquigarrow$ EOM

Conserved Quantities:

- For a Killing vector field ξ^μ: $E_\xi = p_\mu \xi^\mu + \frac{1}{2} S^{\mu\nu} \nabla_\mu \xi_\nu$
- Neglecting $J^{\mu\nu\alpha\beta}$ etc.: mass $\underline{m} := \sqrt{-p_\mu p^\mu}$ or $m := -u^\mu p_\mu$ (SSC dep.)
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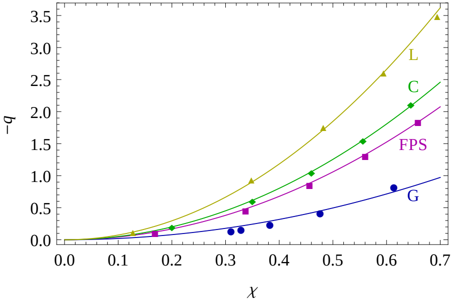
Quadrupole Deformation due to Spin

for neutron stars: Laarakkers, Poisson (1997)

- Here $m = 1.4 M_\odot$
- Dim.-less mass quadrupole: q
- Dim.-less spin: χ
- Quadratic fit is extremely good:

$$-q \approx C_{ES^2} \chi^2$$

- $C_{ES^2} = 4.3 \ldots 7.4$, EOS dependent
- Also depends on mass
- For black holes $C_{ES^2} = 1$



see Laarakkers, Poisson gr-qc/9709033

- higher multipoles: Pappas, Apostolatos (2012)

Quadrupole Action

see Porto, Rothstein (2008)

$$R_M = -\mu \underbrace{\sqrt{u_\mu u^\mu}}_{=:u} + \underbrace{\frac{1}{\mu u} B_{\mu\nu} S^\mu u_\alpha S^{\alpha\nu}}_{\text{SSC preserving}} + \underbrace{\frac{C_{ES^2}}{2\mu u} E_{\mu\nu} S^\mu{}_\alpha S^{\alpha\nu}}_{\text{deformation due to spin}} + \dots$$

$$E_{\mu\nu} \sim R_{\mu\alpha\nu\beta} u^\alpha u^\beta \qquad B_{\mu\nu} \sim \tfrac{1}{2} \epsilon_{\mu\rho\alpha\beta} R_{\nu\sigma}{}^{\alpha\beta} u^\rho u^\sigma \qquad S^\mu = \tfrac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu S_{\alpha\beta}$$

- μ and C_{ES^2}: constants, matched to single object
- Notice: $\mu \neq m \neq \underline{m}$
- Connection to Dixon's EOM: Bailey, Israel (1975)

$$J^{\mu\nu\alpha\beta} = -6 \frac{\partial L}{\partial R_{\mu\nu\alpha\beta}}$$

- Covariant mass quadrupole: (for $u = 1$)

$$Q^{\mu\nu} \sim 2 \frac{\partial L}{\partial E_{\mu\nu}} = \frac{C_{ES^2}}{\mu} S^\mu{}_\alpha S^{\alpha\nu}$$

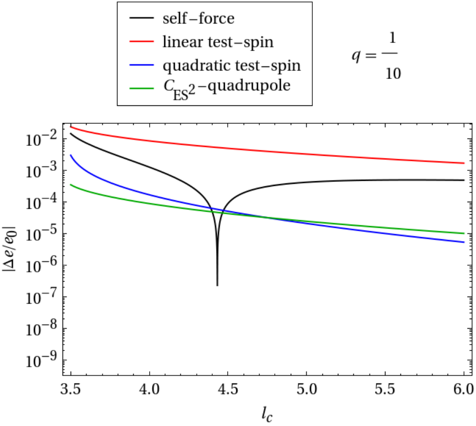
Motion in Schwarzschild background

Steinhoff, Puetzfeld (2012); there also tidal deformation and Kerr background are covered

- Conserved quantities:
  E_{∂_t}, E_{∂_ϕ}, S, μ
- Circular orbits, aligned spin
- SSC: $S^{\mu\nu} p_\nu = 0$

$\Rightarrow p_\nu$, $S^{\mu\nu}$ fixed algebratically!

- Binding energy:
  $e(l_c) = E_{\partial_t}/\mu - 1$
- Orbital angular momentum: l_c



spin effects for $\hat{a}_2 = 1$, $C_{ES^2} = 1$

- Taylor-expansion: $e(l_c) = e_0(l_c) + e_1(l_c) + e_2(l_c) + \ldots$
- Scaling: $e_1 \propto q\hat{a}_2$, $e_2^{S^2} \propto -q^2\hat{a}_2^2$, $e_2^{C_{ES^2}} \propto -C_{ES^2} q^2 \hat{a}_2^2$
- Spin-induced quadrupole effects scale like second-order self-force ($\sim q^2$)

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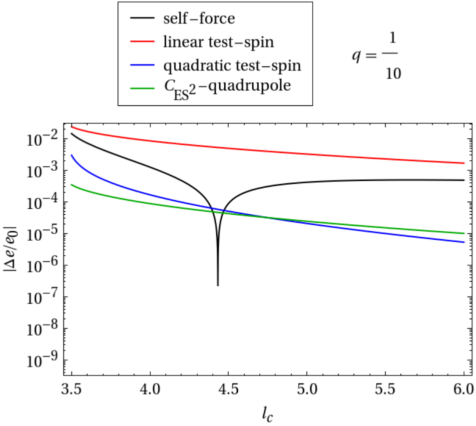

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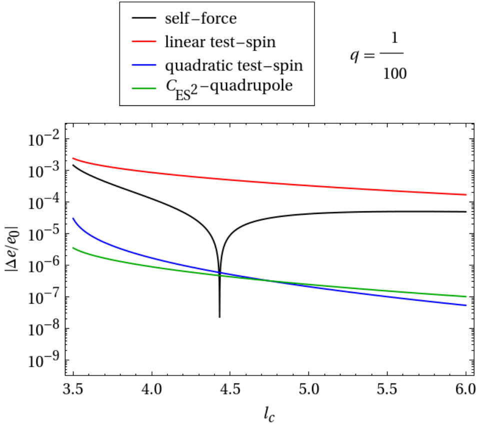

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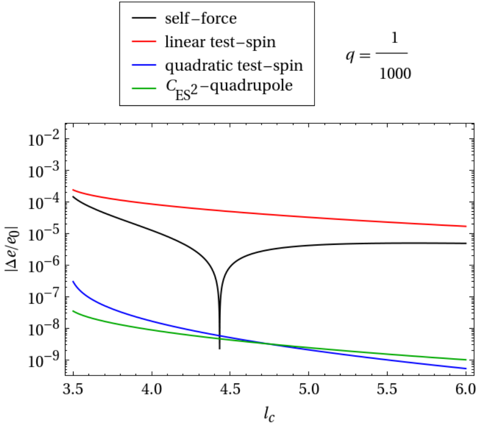

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Tidal deformation: Newtonian case revised

in preparation, with S. Chakrabarti and T. Delsate

Motivation:

- Adiabatic tidal effects may not be sufficient [Maselli et.al. (2012)]
- Resonances of orbital motion and oscillation modes of the object
- Better understanding of tidal interactions

$$L = \frac{1}{2} m \dot{\mathbf{R}}_*^2 - m\Phi(\mathbf{R}_*) - \frac{1}{2} Q^{ij} \partial_i \partial_j \Phi(\mathbf{R}_*) + \ldots$$

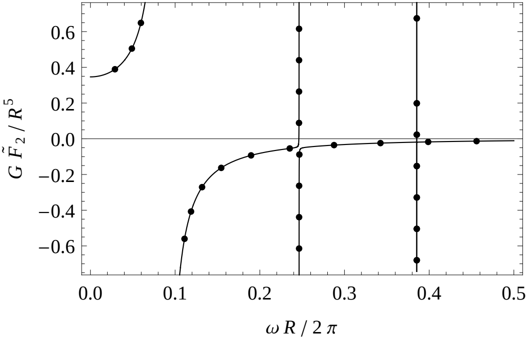
- Idea: response function for Q^{ij} [Goldberger, Rothstein, hep-th/0511133]

$$Q^{ij}(t) = -\frac{1}{2} \int dt' \, F^{ij}_{kl}(t, t') \Phi_{,kl}(t')$$

- From Newtonian tidal theory:

$$F(\omega) = \sum_n \frac{I_n^2}{\omega_n^2 - \omega^2}$$

- ω_n are the mode frequencies
- I_n related to overlap integrals



response function $F(\omega)$ for the quadrupole Q^{ij}

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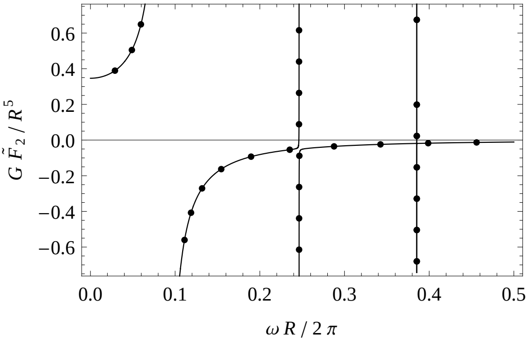

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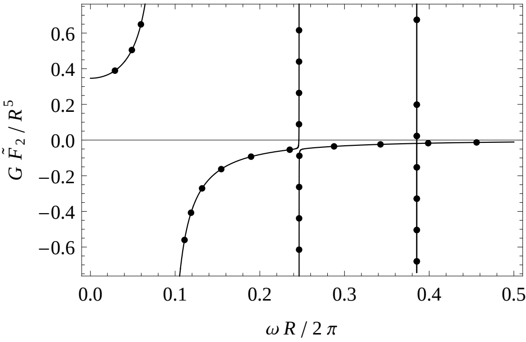

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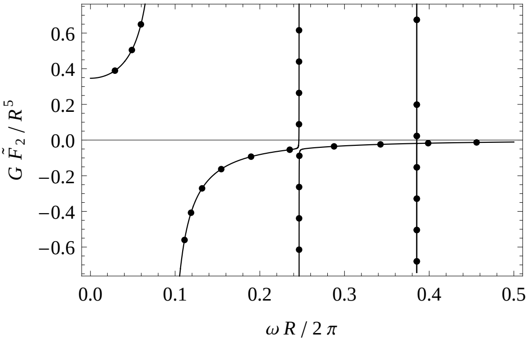

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Thank you for your attention

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Térence Delsate
Dirk Puetzfeld

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