Action principles for extended bodies

in gravitational and electromagnetic fields

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Outline

Action for multipoles

- Charge/mass density ρ centered around $\mathbf{z}$, external field ϕ_{ext}
- Can Taylor expand:

$$\phi_{\text{ext}} = \phi_{\text{ext}}(\mathbf{z}) + \partial_i \phi_{\text{ext}}(\mathbf{z}) x^i + \frac{1}{2} \partial_i \partial_j \phi_{\text{ext}}(\mathbf{z}) x^i x^j + \ldots$$

- $\Delta \phi_{\text{ext}} = 0$ around $\mathbf{z}$: Make it tracefree, $\hat{x}^{K_l} = [x^{k_1} x^{k_2} \ldots x^{k_l}]^{\text{STF}}$

$$\phi_{\text{ext}} = \sum_l \frac{1}{l!} \partial_{K_l} \phi_{\text{ext}}(\mathbf{z}) \hat{x}^{K_l}$$

- In matter-field-interaction Lagrangian:

$$L_{\text{int}} = - \int d^3x \, \rho \phi_{\text{ext}} = - \sum_l \frac{1}{l!} Q^{K_l} \partial_{K_l} \phi_{\text{ext}}(\mathbf{z})$$

- Multipole moments:

$$Q^{K_l} = \int d^3x \, \rho \hat{x}^{K_l}$$

- Examples:

$$L_{\text{el. dipole}} = -Q_e^k F_{0k}, \qquad L_{\text{gr. quadrupole}} = -\frac{1}{2} Q_g^{kl} R_{0k0l}$$

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Generic Lagrangian

see e.g. Bailey, Israel, Commun. math. Phys. **42** 65 (1975)

- Generic worldline Lagrangian:

$$L_M = L_M(u^\mu, \Omega^{\mu\nu}, g_{\mu\nu}(z^\alpha), R_{\mu\nu\rho\sigma}(z^\alpha), F_{\mu\nu}(z^\alpha), \dots)$$

- Multipole contribution to center-of-mass motion:

$$\delta L_M = \dots - \frac{1}{6} J^{\nu\rho\beta\alpha} \nabla_\mu R_{\nu\rho\beta\alpha} \delta z^\mu - \frac{1}{2} D^{\alpha\beta} \nabla_\mu F_{\alpha\beta} \delta z^\mu$$

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- Multipole contribution to spin motion:

$$\frac{DS^{\mu\nu}}{d\tau} = 2 S_\alpha{}^{[\mu} \Omega^{\nu]\alpha} = 2 p^{[\mu} u^{\nu]} + \frac{4}{3} R^{[\mu}{}_{\rho\alpha\beta} J^{\nu]\rho\alpha\beta} + 2 D^{\alpha[\mu} F^{\nu]}{}_\alpha$$

- Covariance of L_M:

$$-G^{\nu\alpha} g_{\mu\alpha} + p_\mu u^\nu + S_{\mu\alpha} \Omega^{\nu\alpha} + \frac{2}{3} J^{\nu\rho\alpha\beta} R_{\mu\rho\alpha\beta} + D^{\nu\alpha} F_{\mu\alpha} = 0$$

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Equations of motion

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$$\frac{Dp_\mu}{d\tau} = 0 - \frac{1}{2} R_{\mu\rho\beta\alpha} u^\rho S^{\beta\alpha} - \frac{1}{6} \nabla_\mu R_{\nu\rho\beta\alpha} J^{\nu\rho\beta\alpha} - \frac{1}{2} \nabla_\mu F_{\alpha\beta} D^{\alpha\beta} + \dots$$

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$$\frac{DJ^{\mu\nu\alpha\beta}}{d\tau} = ???, \quad \frac{D\,D^{\mu\nu}}{d\tau} = ???$$

- Geodesic equation: momentum p_μ
- Mathisson (1937), Papapetrou (1951): spin / dipole $S^{\mu\nu}$
- Dixon (∼1974): quadrupole $J^{\mu\nu\alpha\beta}$, ...

Traditional approach: $T^{\mu\nu}{}_{;\nu} = 0 \rightsquigarrow$ EOM
That is, EOM for p_μ and $S^{\mu\nu}$ follow from generic principles!

Action approach: assume generic covariant action
Simple, but more restrictive. Still the resulting EOM have the same form!

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Finite size effects in electrodynamics

Galley, Leibovich, Rothstein, PRL **105** 094802 (2010)

- Effective action: $a^\mu = \frac{du^\mu}{d\tau}, \qquad u = \sqrt{-u_\rho u^\rho}$

$$S = \int d\tau \left[m + e u^\mu A_\mu + C_1 a_\mu a^\mu + C_2 \frac{u_\mu a_\nu}{u} F^{\mu\nu} \right]$$

- For spherical symmetric charged shell:

$$C_1 = \frac{2}{9} e^2 R, \qquad C_2 = \frac{1}{6} e R^2$$

- Can be used to calculate corrections to the Abraham-Lorentz-Dirac force
- Polarization effects:

$$S_{\text{polar}} = \int d\tau \left[C_E R^3 F_{\mu\nu} F^{\mu\nu} + C_B R^3 u^\mu u^\nu F_{\mu\rho} F^\rho{}_\nu \right]$$

- Induced dipole:

$$D^{\mu\nu} = -2 \frac{\partial L_M}{\partial F_{\mu\nu}} = -4 C_E R^3 F^{\mu\nu} - 4 C_B R^3 u^{[\mu} F^{\nu]\rho} u_\rho$$

Quadrupole Effective Action

see e.g. Porto, Rothstein (2008); Goldberger, Rothstein (2006)

$$L_{\text{quad}} = \underbrace{\frac{1}{mu} B_{\mu\nu} S^{\mu} u_{\alpha} S^{\alpha\nu}}_{\text{SSC preserving}} + \underbrace{\frac{C_{ES^2}}{2 m_c u} E_{\mu\nu} S^{\mu}{}_{\alpha} S^{\alpha\nu}}_{\text{deformation due to spin}} + \underbrace{\frac{\mu_2}{4u^3} E_{\mu\nu} E^{\mu\nu}}_{\text{tidal deformation}} + \dots$$

$$E_{\mu\nu} \sim R_{\mu\alpha\nu\beta} u^{\alpha} u^{\beta} \qquad B_{\mu\nu} \sim \tfrac{1}{2} \epsilon_{\mu\rho\alpha\beta} R_{\nu\sigma}{}^{\alpha\beta} u^{\rho} u^{\sigma} \qquad S^{\mu} = \tfrac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_{\nu} S_{\alpha\beta}$$

- m, C_{ES^2}, and μ_2: constants, matched to single object
- From Bailey, Israel (1975):

$$J^{\mu\nu\alpha\beta} = -6 \frac{\partial L}{\partial R_{\mu\nu\alpha\beta}}$$

- Covariant mass quadrupole: (for $u = 1$)

$$\text{mass quadrupole} \sim 2 \frac{\partial L}{\partial E_{\mu\nu}} = \frac{C_{ES^2}}{m_c} S^{\mu}{}_{\alpha} S^{\alpha\nu} + \mu_2 E^{\mu\nu}$$

Tidal Quadrupole Deformation

for NS, e.g. Hinderer & Flanagan (2008); Damour, Nagar (2009); Binnington, Poisson (2009)

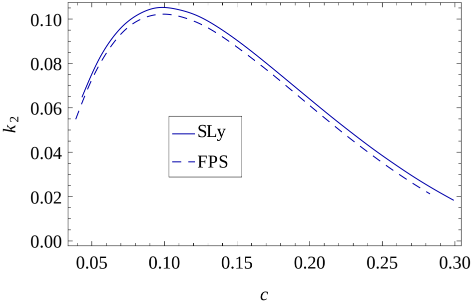
- Linear NS perturbation, thus:

$$-Q = \mu_2 E$$

- Tidal force E (curvature)
- Dim.-less 2nd Love number k_2:

$$k_2 = \frac{3}{2}\frac{\mu_2}{R^5}$$

- Measure for grav. polarizability
- Compactness $c = \dfrac{Gm}{R}$



see Damour, Nagar arXiv:0906.0096

- For certain realistic EOS it holds $k_2 \approx 0.17 - 0.52c$
- For black holes $k_2 = 0$

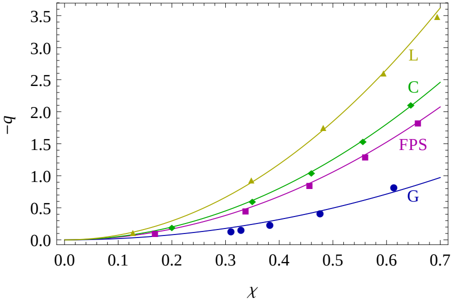
Quadrupole Deformation due to Spin

for neutron stars: Laarakkers, Poisson ApJ **512** 282 (1999)

- Here $m = 1.4 M_\odot$
- Dim.-less mass quadrupole: q
- Dim.-less spin: χ
- Quadratic fit is quite good:

$$-q \approx C_{ES^2} \chi^2$$

- $C_{ES^2} = 4.3 \dots 7.4$, EOS dependent
- Also depends on mass
- For black holes $C_{ES^2} = 1$



see Laarakkers, Poisson (1999)

- RNS code by N. Stergioulas publicly available
- higher multipoles: Pappas, Apostolatos, PRL **108** 231104 (2012)

Application to test-particle motion

Conserved Quantities:

- For a Killing vector field ξ^μ: $E_\xi = p_\mu \xi^\mu + \frac{1}{2} S^{\mu\nu} \nabla_\mu \xi_\nu$
- Neglecting $J^{\mu\nu\alpha\beta}$ etc.: mass $\underline{m} := \sqrt{-p_\mu p^\mu}$ or $\overline{m} := -u^\mu p_\mu$ (SSC dep.)
  spin-length $S = \sqrt{\frac{1}{2} S_{\mu\nu} S^{\mu\nu}}$

Method to construct simple solutions to the EOM:

- Conserved quantities for stationary axisymmetric spacetime:
  E_{∂_t}, E_{∂_ϕ}, S, $\underline{m}$
- Assume Circular orbits and aligned spin
- Spin supplementary condition: $S^{\mu\nu} p_\nu = 0$

$\Rightarrow p_\nu$, $S^{\mu\nu}$ fixed algebratically!

Literature

S. N. Rasband, PRL **30** 111 (1973) [spinning particle in Kerr]
K. P. Tod, F. de Felice, and M. Calvani, Nuovo Cim. B **34** 365 (1976)
S. Suzuki and K. Maeda, PRD **58** 023005 (1998) [investigation of chaos]
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R. Hojman and S. Hojman, PRD **15** 2724 (1977) [spin+charge in Kerr-Newman]
J. Steinhoff, D. Puetzfeld, PRD **86** 044033 (2012) [spin+quadrupole in Kerr]

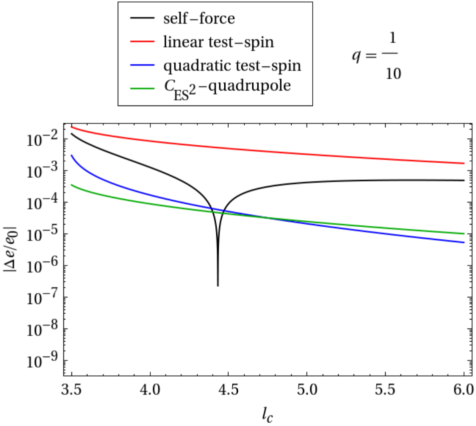
Motion in Schwarzschild background

Steinhoff, Puetzfeld, PRD **86** 044033 (2012)
similar quadrupole model: Bini, Geralico PRD **87** 024028 (2013)

Need conserved quantities:
E_{∂_t}, E_{∂_ϕ}, S, m

- E_{∂_t} and E_{∂_ϕ} still conserved
- S conserved due to symmetry of action
- m constant parameter in action

- Binding energy:
$$e(l_c) = E_{\partial_t}/m - 1$$
- Orbital angular momentum: l_c



spin effects for $\hat{a}_2 = 1$, $C_{ES^2} = 1$

- Multipole expansion: $e(l_c) = e_0(l_c) + e_1(l_c) + e_2(l_c) + \ldots$
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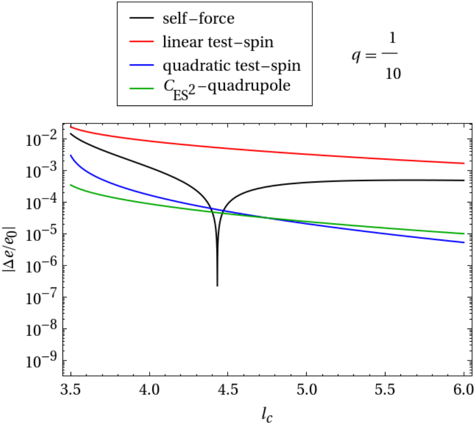

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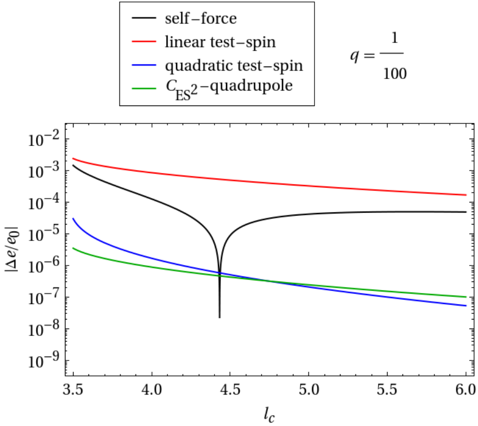

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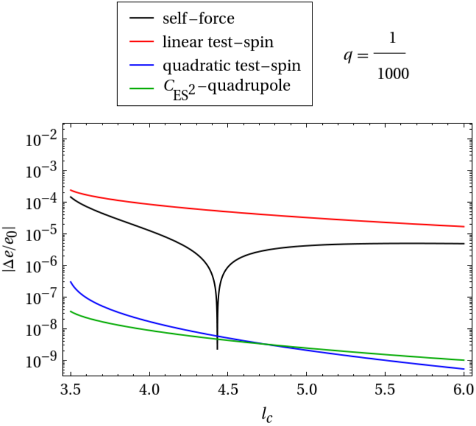

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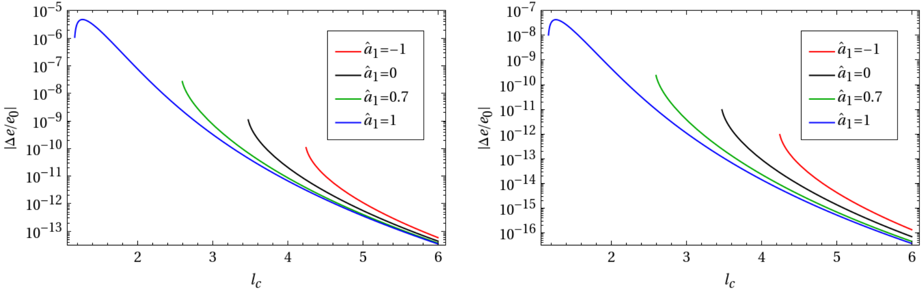
Results for Kerr background

Steinhoff, Puetzfeld, PRD **86** 044033 (2012)

tidal effects for neutron stars and mass ratio $q = \frac{1}{50}$

($k_2 = 0.1$, $j_2 = -0.01$, $\hat{R} = 5$)



gravito-electric tidal effects — gravito-magnetic tidal effects

- Scaling: $e_2^{k_2} \propto -k_2 q^4 \hat{R}^5$ $\qquad e_2^{j_2} \propto j_2 q^4 \hat{R}^5$
- For $\hat{a}_1 = 1$ circular orbits are possible at the horizon!
- Limit due to tidal disruption: $\frac{e_2^{k_2}}{e_0} \lesssim \frac{k_2}{4\hat{R}} \sim 10^{-2}$

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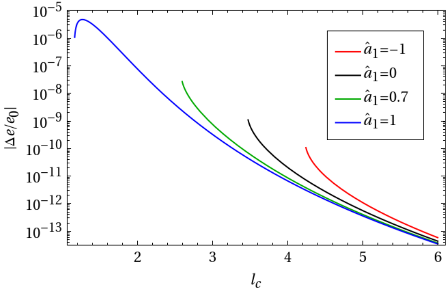

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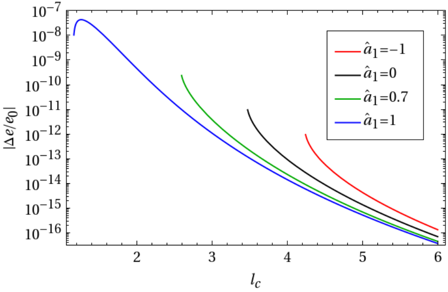

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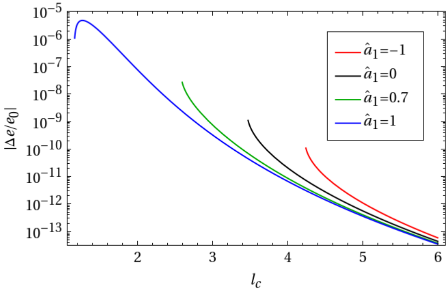

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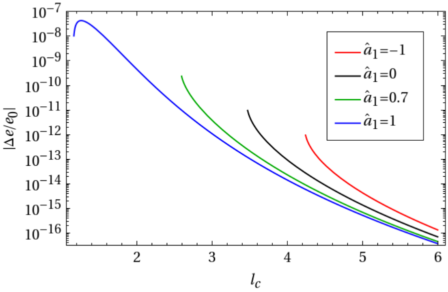

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Outline

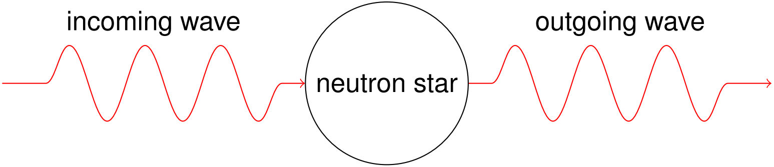
Tidal polarization beyond the adiabatic case

Chakrabarti, Delsate, Steinhoff, arXiv:1304.2228 [gr-qc]

- Adiabatic tidal effects may not be sufficient [Maselli et.al. (2012)]
- Resonances of orbital motion and oscillation modes of the object
- Idea: response function for Q^{ab} [Goldberger, Rothstein, hep-th/0511133]

$$Q^{ab}(t) = -\frac{1}{2} \int dt' \; F^{ab}{}_{cd}(t, t') \, E^{cd}(t')$$

- Analysis in Fourier space:



- Analogy to optics: refractive index is response, need phase shift
  also: absorption from imaginary part of $F(\omega)$

Methods and results

Chakrabarti, Delsate, Steinhoff, arXiv:1304.2228 [gr-qc]

- Method: inhomogeneous Regge-Wheeler equation

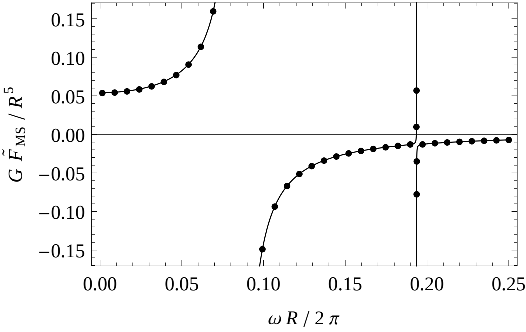
$$\frac{d^2X}{dr_*^2} + \left[\left(1 - \frac{2M}{r}\right)\frac{l(l+1) - \frac{6M}{r}}{r^2} + \omega^2\right] X = S,$$

- Analytic solutions for hom. equation are known: series of ${}_1F_1$ and ${}_2F_1$ [Mano, Suzuki, Takasugi, arXiv:gr-qc/9605057]

- Fit for the response:

$$F(\omega) = \sum_n \frac{q_n^2}{\omega_n^2 - \omega^2}$$

- Just like in Newtonian case!
- ω_n are the mode frequencies
- q_n related to overlap integrals
- Matching scale is fitted, too



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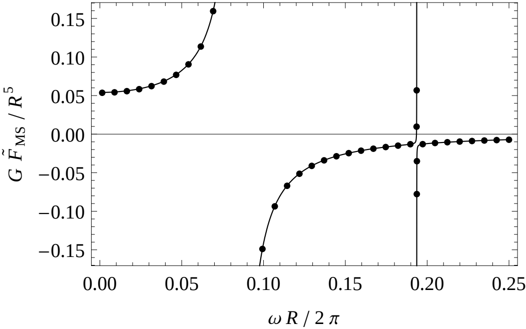

Outline

Conclusions

Action principles for extended compact objects:

- Restricted, but applicable to many physical situations
- Simple compared to standard approaches
- Easy to identify conserved quantities $\rightarrow$ solutions to EOM
- Straightforward to extend field content, e.g., include $F^{\mu\nu}$

Future work on dynamic multipoles and tides:

- More realistic NS models: rotation, crust, . . .
- Resonances with orbital motion
- Instabilities of modes, shattering of crust, connection to GRB, . . .

Thank you for your attention

and special thanks to my collaborators

Sayan Chakrabarti
Térence Delsate
Dirk Puetzfeld
Gerhard Schäfer

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