Canonical formulation of spinning objects

in General Relativity from an action approach

Jan Steinhoff Steven Hergt Gerhard Schäfer

since 1558

Theoretisch-Physikalisches Institut
Friedrich-Schiller-Universität Jena

GR19 Conference, Session B3, July 5th, 2010, Mexico City

since 1558

DFG: SFB/TR7 “Gravitational Wave Astronomy” and GRK 1523

(3+1)-Decomposition

- Normal vector:

$$n_\mu = (-N, 0, 0, 0)\,, \quad n^\mu = \frac{1}{N}(1, -N^i)\,, \quad n_\mu n^\mu = -1$$

- Projector:

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}, \quad g_{ij} = \gamma_{ij}\,, \quad \gamma_{ik}\gamma^{kj} = \delta_{ij}$$

- Extrinsic curvature:

$$K_{ij} \equiv -n_{(i||j)}$$

$$\pi^{ij} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{lj} - \gamma^{ij}\gamma^{kl})K_{kl}$$

Point-Mass Action in ADM Form

$$W = \int dt\, p_i \dot{z}^i + \int d^4x \left[\frac{1}{16\pi} \pi^{ij} \gamma_{ij,0} - H \right]$$

$$H = \int d^3x (N\mathcal{H} - N^i \mathcal{H}_i) + E[\gamma_{ij}]$$

- Constraint equations:

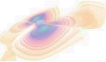
$$0 = \mathcal{H} \equiv -\frac{1}{16\pi\sqrt{\gamma}} \left[\gamma \mathrm{R} + \frac{1}{2} \left(\gamma_{ij} \pi^{ij} \right)^2 - \gamma_{ij} \gamma_{kl} \pi^{ik} \pi^{jl} \right] + \mathcal{H}^{\text{matter}}$$

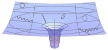
$$0 = \mathcal{H}_i \equiv \frac{1}{8\pi} \gamma_{ij} \pi^{jk}{}_{;k} + \mathcal{H}_i^{\text{matter}}$$

- Source terms are related to the stress-energy tensor $T^{\mu\nu}$:

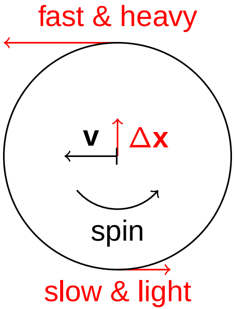
$$\mathcal{H}^{\text{matter}} \equiv \sqrt{\gamma} T_{\mu\nu} n^\mu n^\nu = \sqrt{m^2 + \gamma^{ij} p_i p_j}\, \delta$$

$$\mathcal{H}_i^{\text{matter}} \equiv -\sqrt{\gamma} T_{i\nu} n^\nu = p_i \delta$$





Spin in Special Relativity



- Spin is a 4-tensor $S^{\mu\nu}$:
  - Spin is $S^{ij} = \varepsilon^{ijk} S_k$.
  - Mass dipole related to S^{i0}.
- Different mass centers.
- Need spin supplementary condition:
  - Møller SSC: $\tilde{S}^{\mu 0} = 0$
  - Covariant SSC: $S^{\mu\nu} p_\nu = 0$
  - Newton-Wigner (canonical) SSC: $m\hat{S}^{\mu 0} + \hat{S}^{\mu\nu} p_\nu = 0$

Non-Relativistic Spherical Top

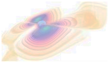
without center-of-mass motion

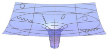
- Body fixed coordinates $z^{[i]}$: $\quad z^i(t) = R_{[j]i}(t) z^{[j]}, \quad R_{[k]i} R_{[k]j} = \delta_{ij}$
- Angular velocity tensor: $\quad \Omega^{ij} = \varepsilon_{ijk} \Omega^k = R_{[k]i} \dot{R}_{[k]j}$
- Spin tensor: $\quad S_{ij} = 2 \frac{\partial L}{\partial \Omega^{ij}}$
- Legendre transformed:

$$L = \frac{1}{2} S_{ij} \Omega^{ij} - H[R_{ij}, S_{ij}]$$

- Usual Poisson brackets for the spin:

$$\{\hat{S}_i, \hat{S}_j\} = \varepsilon_{ijk} \hat{S}_k$$

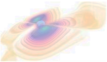


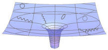


Relativistic Spherical Top

E.g., Hanson and Regge (1974)

- No rigid bodies.
- Mathematical abstraction: Top is
  - Worldline with Lorentz-matrix $\Lambda_{A\mu}$: $\eta^{AB}\Lambda_{A\mu}\Lambda_{B\nu} = \eta_{\mu\nu}$
  - $\Lambda_{A\mu}$ is pure rotation in rest-frame: $\Lambda^{[i]\mu} p_{\mu} = 0$
- Angular velocity tensor: $\Omega^{\mu\nu} = \Lambda_{A}{}^{\mu} \dfrac{d\Lambda^{A\nu}}{d\tau}$
- Spin tensor: $S_{\mu\nu} = 2\dfrac{\partial L}{\partial \Omega^{\mu\nu}}$
- Associated SSC: $S_{\mu\nu} p^{\mu} = 0$





Minimal Coupling

- Variate Λ^{Aa} and tetrad $e_{a\mu}$:

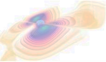
$$\Lambda_{Aa}\Lambda^{A}{}_{b} = \eta_{ab} \quad \leftrightarrow \quad \delta\theta_{ab} = -\delta\theta_{ab} = \Lambda^{C}{}_{a}\delta\Lambda_{Cb}$$

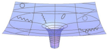
- Matter Lagrangian density and constraints:

$$\mathscr{L}_M = \int d\tau \left[p_\mu u^\mu + \frac{1}{2} S_{ab}\Omega^{ab} \right] \delta_{(4)}$$

$$\Omega^{ab} = \Lambda_A{}^a \frac{D\Lambda^{Ab}}{d\tau} = \Lambda_A{}^a \left[\frac{d\Lambda^{Ab}}{d\tau} - \Lambda^A{}_c \omega_\mu{}^{cb} u^\mu \right]$$

$$S_{ab}p^b = 0\,, \quad \Lambda^{[i]a} p_a = 0\,, \quad p_\mu p^\mu + m^2 = 0$$





Result of Variation

- Approximated linear in spin.
- Field equations with stress-energy tensor (Mathisson, Tulczyjew):

$$\sqrt{-g}\; T^{\mu\nu} = \int d\tau \left[m u^{\mu} u^{\nu} \delta_{(4)} - \left(S^{\alpha(\mu} u^{\nu)} \delta_{(4)} \right)_{||\alpha} \right]$$

- EOM (Mathisson, Papapetrou):

$$\frac{DS^{\mu\nu}}{d\tau} = 0\,, \qquad \frac{Dp_{\mu}}{d\tau} = \frac{1}{2} S^{\lambda\nu} u^{\gamma} R_{\mu\gamma\nu\lambda}$$

Reduction of the Matter Part

- Solve matter constraints, Schwinger time gauge $e_{(0)\mu} = n_\mu$, $\tau = t$.
- Variable redefinitions:

$$z^i = \hat{z}^i - \frac{nS^i}{m - np}, \quad np = -\sqrt{m^2 + \gamma^{ij} p_i p_j}$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i nS_j}{m - np} + \frac{p_j nS_i}{m - np}, \quad nS_i = -\frac{p_k \gamma^{kj} \hat{S}_{ji}}{m}$$

$$\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)} \left(\delta_{kj} + \frac{p_{(k)} p_{(j)}}{m(m - np)} \right), \quad \gamma_{ik} \gamma_{jl} A^{kl} = \frac{1}{2} \hat{S}_{ij} + \frac{m p_{(i} nS_{j)}}{np(m - np)}$$

$$\hat{p}_i = p_i + K_{ij} nS^j + A^{kl} e_{(j)k} e^{(j)}_{l,i} - \left(\frac{1}{2} S_{kj} + \frac{p_{(k} nS_{j)}}{np} \right) \Gamma^{kj}{}_i$$

- Matter Lagrangian density now, with $\hat{\Omega}^{(i)(j)} = \hat{\Lambda}_{[k]}{}^{(i)} \dot{\hat{\Lambda}}^{[k](j)}$,

$$\mathscr{L}_M = A^{ij} e_{(k)i} e^{(k)}_{j,0} \hat{\delta} + \hat{p}_i \dot{\hat{z}}^i \hat{\delta} + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} \hat{\delta} - N \mathscr{H}^{\text{matter}} + N^i \mathscr{H}_i^{\text{matter}}$$

ADM Formalism with Spin

- Legendre transformation for gravitational field.
- Spatial symmetric gauge (Kibble 1963): $e_{(i)j} = e_{ij} = e_{ji}$

$$e_{ij} e_{jk} = \gamma_{ik} \quad \Rightarrow \quad (e_{ij}) = \sqrt{(\gamma_{ij})}$$

- Action:

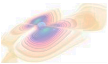
$$W = \frac{1}{16\pi} \int d^4 x \hat{\pi}^{ij} \gamma_{ij,0} + \int dt \left[\hat{p}_i \dot{\hat{z}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} - H \right]$$

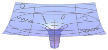
$$H = \int d^3 x (N \mathscr{H} - N^i \mathscr{H}_i) + E[\gamma_{ij}]$$

- Canonical field momentum:

$$\hat{\pi}^{ij} = \pi^{ij} + 8\pi A^{(ij)} \hat{\delta} + 16\pi B^{ij}_{kl} A^{[kl]} \hat{\delta}$$

$$2B^{kl}_{ij} = e_{mi} \frac{\partial e_{mj}}{\partial \gamma_{kl}} - e_{mj} \frac{\partial e_{mi}}{\partial \gamma_{kl}}$$



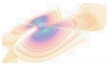


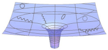
Next-to-Leading Order (NLO) Spin-Orbit

See also: Tagoshi, Ohashi, Owen (2001); Faye, Blanchet, Buonanno (2006)
Hamiltonian first derived: Damour, Jaranowski, and Schäfer (2008)

$$
\begin{aligned}
H_{SO}^{NLO} = & -\frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2}\left[\frac{5m_2\hat{\mathbf{p}}_1^2}{8m_1^3} + \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)}{4m_1^2} - \frac{3\hat{\mathbf{p}}_2^2}{4m_1m_2} \right. \\
& \left. + \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{4m_1^2} + \frac{3(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})^2}{2m_1m_2}\right] \\
& + \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2}\left[\frac{(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)}{m_1m_2} + \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{m_1m_2}\right] \\
& + \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{p}}_2)}{\hat{r}_{12}^2}\left[\frac{2(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{m_1m_2} - \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})}{4m_1^2}\right] \\
& - \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3}\left[\frac{11m_2}{2} + \frac{5m_2^2}{m_1}\right] \\
& + \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3}\left[6m_1 + \frac{15m_2}{2}\right] + (1 \leftrightarrow 2)
\end{aligned}
$$

since 1558



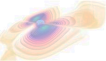


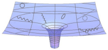
NLO Spin$_1$-Spin$_2$

Partial result: Porto and Rothstein (2006)

$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_1S_2} = {} & \frac{1}{2m_1m_2\hat{r}_{12}^3}\Big[\tfrac{3}{2}((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12})+\tfrac{1}{2}(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)\\
& +6((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12})-\tfrac{1}{2}(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_1)\\
& -15(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})+(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_2)\\
& -3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)+3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})\\
& +3(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})+3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})\\
& +3(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})-3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})\Big]\\
& +\frac{3}{2m_1^2\hat{r}_{12}^3}\Big[-((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12})\\
& +(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})^2-(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})\Big]\\
& +\frac{3}{2m_2^2\hat{r}_{12}^3}\Big[-((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})\\
& +(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})^2-(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})\Big]\\
& +\frac{6(m_1+m_2)}{\hat{r}_{12}^4}\Big[(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)-2(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})\Big]
\end{aligned}
$$

since 1558





Higher Orders and Radiation Reaction

- Next-to-next-to-leading order (J. Hartung):
  - Very similar to calculation of the 3PN point-mass Hamiltonian.
  - Requires dimensional regularization.
- Radiation reaction from interaction Hamiltonian (with H. Wang):

$$H^{\text{int}} = \frac{1}{16\pi}\int d^3x \Big[\left(B_{(4)ij} + B_{(6)ij}\right) h^{\text{TT}}_{ij} - \frac{16\pi}{8}\mathscr{H}^{\text{matter}}_{(2)} \left(h^{\text{TT}}_{ij}\right)^2 - \frac{1}{4}\phi_{(2)}\left(h^{\text{TT}}_{ij,k}\right)^2 + 2(V^i_{(3)}\phi_{(2),j} - \tilde{\pi}^{ij}_{(5)\text{matter}})\hat{\pi}^{ij\text{TT}} \Big]$$

$$B_{(4)ij} = 16\pi \frac{\delta\left(\int d^3x\, \mathscr{H}^{\text{matter}}_{(8)}\right)}{\delta h^{\text{TT}}_{ij}} - \frac{1}{8}\phi_{(2),i}\phi_{(2),j}$$

$$B_{(6)ij} = 16\pi \frac{\delta\left(\int d^3x \left(\mathscr{H}^{\text{matter}}_{(10)} - \frac{1}{4}\mathscr{H}^{\text{matter}}_{(8)}\phi_{(2)} + 2\mathscr{H}^{\text{matter}}_{(7)k} V^k_{(3)}\right)\right)}{\delta h^{\text{TT}}_{ij}} + \frac{1}{4}\phi_{1(4)}\phi_{(2),ij} + \frac{3}{8}\phi_{2(4)}\phi_{(2),ij} + \frac{5}{64}\phi_{(2)}\phi_{(2),i}\phi_{(2),j} + 2\tilde{\pi}^{jk}_{(3)}\left(\tilde{\pi}^k_{(3),i} - \tilde{\pi}^i_{(3),k}\right) + 2\tilde{\pi}^{ij}_{(3),k} V^k_{(3)} + \frac{1}{2}\tilde{\pi}^{ij}_{(3)}\tilde{\pi}^k_{(3),k}$$

Thank you for your attention

and the German Research Foundation DFG for support