The PN Approximation beyond Point-Masses

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The post-Newtonian (PN) Approximation

of general relativity (GR)

- Newtonian limit: slow motion and weak field limit of GR
- PN approximation: expansion around Newtonian limit
- Bound binary: Virial theorem

$$2\langle\text{kinetic energy}\rangle_t = -\langle\text{potential energy}\rangle_t$$

$$2\,\tfrac{1}{2}\mu v^2 \sim \frac{GM\mu}{r}$$

- one PN order $\hat{=}\ c^{-2}$
- half orders $\hat{=}\ c^{-1} \leftrightarrow$ antisymmetry under time-reversal $\hat{=}$ radiation
- Assumption on $T^{\mu\nu}$: "strength" decreases as $T^{00} \curvearrowright T^{i0} \curvearrowright T^{ij}$
- Source $T^{\mu\nu}$ approximated by multipoles, parameter $\sim \frac{R_{\text{object}}}{r}$

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Electrostatic Multipoles

- Taylor series of Fourier-transformed charge density ρ:

$$\rho(\mathbf{k}) = \left(q + i q^i k_i + \frac{1}{2!} i^2 q^{ij} k_i k_j + \ldots \right) (2\pi)^{-3/2}$$

- In position space:

$$\rho(\mathbf{x}) = \left(q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \ldots \right) \delta(\mathbf{x})$$

$$\delta(\mathbf{x}) \leftrightarrow (2\pi)^{-3/2} \qquad \partial_i \leftrightarrow -i k_i$$

- The potential ϕ reads:

$$\phi = -4\pi \Delta^{-1} \rho = \left(q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \ldots \right) \frac{1}{|\mathbf{x}|}$$

- Electric monopole, dipole, and quadrupole: q, q^i, q^{ij}
- Multipole approximation breaks down for big $|\mathbf{k}|$ or small $|\mathbf{x}|$ ($\sim R_{\text{object}}$)
- Self-energy UV-divergent: $\int \rho\phi \sim \frac{1}{0}$

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Gravitational Multipoles

and the PN approximation beyond point-masses

$$\sqrt{-g}T^{\mu\nu}(x^\sigma) = \int d\tau \Big[u^{(\mu}p^{\nu)}\delta_{(4)} + \left(u^{(\mu}S^{\nu)\alpha}\delta_{(4)}\right)_{||\alpha}$$
$$+ \frac{1}{3}R_{\alpha\beta\rho}{}^{(\mu}J^{\nu)\rho\beta\alpha}\delta_{(4)} - \frac{2}{3}\left(J^{\mu\alpha\beta\nu}\delta_{(4)}\right)_{||(\alpha\beta)} + \ldots \Big]$$

$$u^\mu = \frac{dz^\mu}{d\tau} \qquad \delta_{(4)} = \delta(x^\sigma - z^\sigma)$$

- Point masses only distinguished by a mass $m = \sqrt{-p_\mu p^\mu}$
- Adding a dipole: Spin
- Higher multipoles: Quadrupole, octupole, ... ("finite size effects")
- Dimensional regularization required for self-gravitating objects

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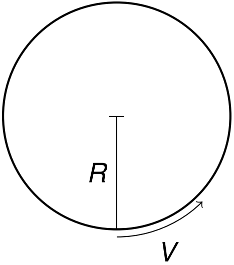
Two Facts on Spin in Relativity

1. Minimal Extension



- ring of radius R and mass M
- spin: $S = R\,M\,V$
- maximal velocity: $V \leq c$
  $\Rightarrow$ minimal extension:

$$R = \frac{S}{MV} \geq \frac{S}{Mc}$$

2. Center-of-mass

- now moving with velocity v
- relativistic mass changes inhom.
- frame-dependent center-of-mass
- need spin supplementary condition,

$$\text{e.g.,} \quad S^{\mu\nu} p_\nu = 0$$

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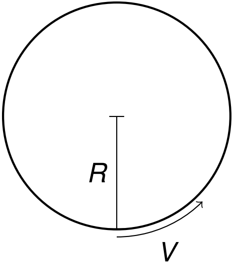

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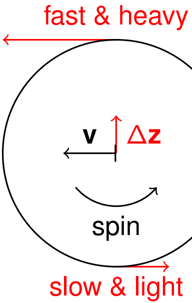

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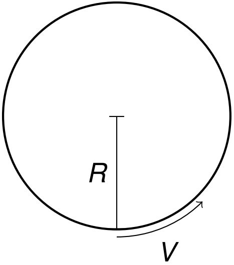

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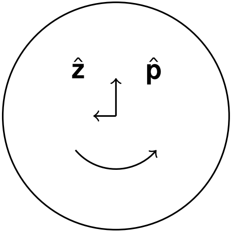

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Angular Velocity and Spin

in Newtonian mechanics and special relativity

	Newton	special relativity
body-fixed frame	$x^i_{\text{bf}} = \Lambda^{ij} x^j$	
rotational degrees of freedom	$\Lambda^{ki}\Lambda^{kj} = \delta_{ij}$	$\eta_{AB}\Lambda^{A\mu}\Lambda^{B\nu} = \eta^{\mu\nu}$
$\hookrightarrow$ supplementary condition		$\Lambda_{i\mu} p^\mu = 0$
Angular Velocity	$\Omega^{ij} = \Lambda^{ki}\frac{\mathrm{d}\Lambda^{kj}}{\mathrm{d}t}$	$\Omega^{\mu\nu} = \Lambda_A{}^\mu \frac{\mathrm{d}\Lambda^{A\nu}}{\mathrm{d}\tau}$
Spin (L: Lagrangian)	$S_{ij} = 2\frac{\partial L}{\partial \Omega^{ij}}$	$S_{\mu\nu} = 2\frac{\partial L}{\partial \Omega^{\mu\nu}}$
$\hookrightarrow$ supplementary condition		$S_{\mu\nu} p^\nu = 0$

Remark:

- Angular velocity vector is $\Omega^i = \frac{1}{2}\epsilon_{ijk}\Omega^{jk}$. Analogous for spin.

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Spin Action in GR

- Minimal coupling:

$$\Omega^{\mu\nu} = \Lambda_A{}^{\mu} \frac{D\Lambda^{A\nu}}{d\tau}$$

$$L = m_c \underbrace{\sqrt{-u_\mu u^\mu}}_{u} + \underbrace{\frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu}}_{\rightsquigarrow \frac{1}{2} S^{ij}\partial_i A_j} + \dots$$

- $m \approx m_c = \text{const}$
- Valid to linear order in spin
- Gravito-magnetic field $A_i \approx -g_{i0}$
- Metric variation problematic:

$$\Lambda_{A\mu}\Lambda^A{}_\nu = g_{\mu\nu} \quad \leftrightarrow \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$$

- Variate Λ^{Aa} and tetrad $e_{a\mu}$: $\quad e_{a\mu}e^a{}_\nu = g_{\mu\nu} \quad \Lambda^A{}_\mu = \Lambda^{Aa}e_{a\mu}$

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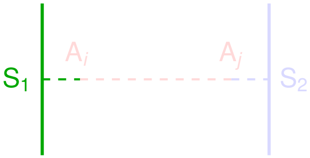
Spin and Gravitomagnetism



$$\int d^3x \, \frac{1}{2} S_1^{ki} \partial_k \delta_1 \; 16\pi G \delta_{ij} \Delta^{-1} \; \frac{1}{2} S_2^{lj} \partial_l \delta_2$$

$$= \int d^3x \, \frac{1}{2} S_1^{ki} \partial_k \delta_1 \, (-2) G S_2^{li} \partial_l \left(\frac{1}{r_2} \right)$$

$$= G S_1^{ki} S_2^{li} \partial_k \partial_l \left(\frac{1}{r_1} \right) \Bigg|_{\mathbf{x}=\mathbf{z}_1}$$

- We will calculate the leading-order $S_1 S_2$ potential
- Here: $\delta_a = \delta(\mathbf{x} - \mathbf{z}_a)$, $r_a = |\mathbf{x} - \mathbf{z}_a|$
- Diagrams encode integrals
- Translation rules: Feynman rules see e.g. Levi arXiv:1006.4139
- Rules follow from the action

Relevance of T^{00}, T^{i0}, T^{ij} revised

N	mass T^{00}	$\rightsquigarrow$ gravito-electric field
1PN	flow T^{i0}	$\rightsquigarrow$ gravito-magnetic field (A_i)
2PN	stress T^{ij}	$\rightsquigarrow$ 3-dim. tensor field

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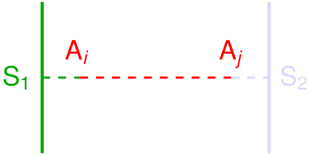

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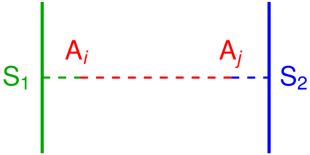

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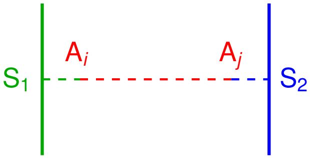

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- Translation rules: Feynman rules see e.g. Levi arXiv:1006.4139
- Rules follow from the action

Relevance of T^{00}, T^{i0}, T^{ij} revised

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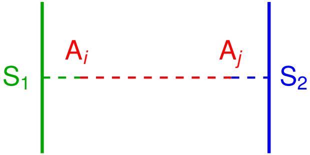

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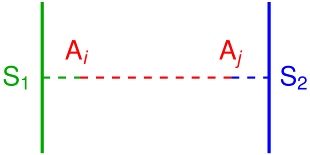

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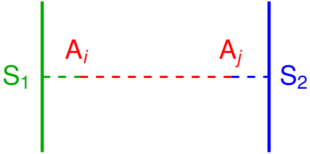

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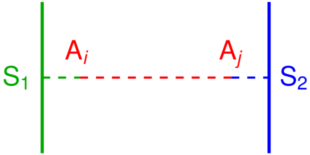

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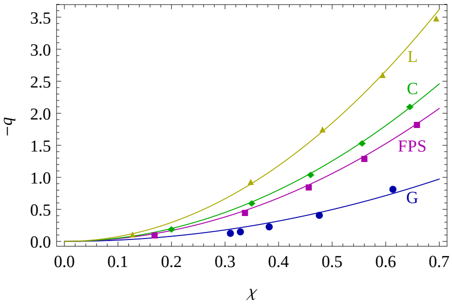
Quadrupole Deformation due to Spin

for neutron stars, Laarakkers, Poisson gr-qc/9709033

- Here $m = 1.4 M_{\odot}$
- Dim.-less mass quadrupole: q
- Dim.-less spin: χ
- Quadratic fit is extremely good:

$$-q \approx C_{ES^2} \chi^2$$

- $C_{ES^2} = 4.3 \dots 7.4$, EOS dependent
- Also depends on mass



see Laarakkers, Poisson gr-qc/9709033

- For black holes $C_{ES^2} = 1$
- S^4-quadrupole is highly suppressed
- RNS code by N. Stergioulas publicly available

Tidal Quadrupole Deformation

for neutron stars, e.g. Damour, Nagar arXiv:0906.0096, Binnington, Poisson arXiv:0906.1366

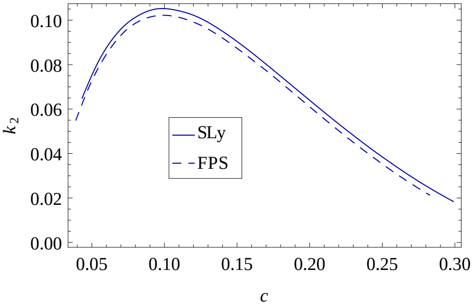
- Linear NS perturbation, thus:

$$-Q = \mu_2 E$$

- Tidal force E (curvature)
- Dim.-less 2nd Love number k_2:

$$k_2 = \frac{3}{2}\frac{\mu_2}{R^5}$$

- Compactness $c = \dfrac{Gm}{R}$



see Damour, Nagar arXiv:0906.0096

- For certain realistic EOS it holds $k_2 \approx 0.17 - 0.52c$
- For black holes $k_2 \sim 0$

Quadrupole Action

see e.g. Porto, Rothstein arXiv:0804.0260, Goldberger, Rothstein hep-th/0511133

$$L_{\text{quad}} = \underbrace{\frac{1}{m_c u} B_{\mu\nu} S^\mu u_\alpha S^{\alpha\nu}}_{\text{SSC preserving}} + \underbrace{\frac{C_{ES^2}}{2m_c u} E_{\mu\nu} S^\mu{}_\alpha S^{\alpha\nu}}_{\text{deformation due to spin}} + \underbrace{\frac{\mu_2}{4u^3} E_{\mu\nu} E^{\mu\nu}}_{\text{tidal deformation}} + \dots$$

$$E_{\mu\nu} \sim R_{\mu\alpha\nu\beta} u^\alpha u^\beta \qquad B_{\mu\nu} \sim \tfrac{1}{2}\epsilon_{\mu\rho\alpha\beta} R_{\nu\sigma}{}^{\alpha\beta} u^\rho u^\sigma \qquad S^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu S_{\alpha\beta}$$

- m_c, C_{ES^2}, and μ_2: constants, matched to single object
- Now: $m_c \neq m$
- From Bailey, Israel (1975):

$$J^{\mu\nu\alpha\beta} = -6 \frac{\partial L}{\partial R_{\mu\nu\alpha\beta}}$$

- Covariant mass quadrupole: (for $u = 1$)

$$\text{mass quadrupole} \sim 2 \frac{\partial L}{\partial E_{\mu\nu}} = \frac{C_{ES^2}}{m_c} S^\mu{}_\alpha S^{\alpha\nu} + \mu_2 E^{\mu\nu}$$

Surface Terms and ADM Hamiltonian

ADM ≙ Arnowitt, Deser, Misner

- Einstein–Hilbert action plus York–Gibbons–Hawking surface term:

$$S_{\text{field}} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}R - \frac{1}{16\pi G}\oint d^3y\,2\sqrt{h}K$$

- ADM energy given by surface integral

$$E_{\text{ADM}} = \frac{1}{16\pi G}\oint d^2s_i[\gamma_{ij,j} - \gamma_{jj,i}]$$

- H^{ADM} ≙ ADM energy E_{ADM} expressed in terms of canonical variables
- Canonical field variables: h_{ij}^{TT}, $\pi^{ij\text{TT}}$ TT ≙ transverse-traceless

DeWitt (1967)

"General relativity is unique among field theories in that its energy may always be expressed as a surface integral."

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Canonical Variables to Linear Order in Spin

- Flat spacetime: Newton-Wigner center and spin are canonical
- Canonical $\hat{z}^i$, $\hat{S}_{ij}$, and $\hat{\Lambda}^{ij}$ are "simple" generalizations of flat space case
- Canonical matter momentum $\hat{p}_i$:

$$p_i = \hat{p}_i + \frac{1}{2}\hat{S}_{kj}\Gamma^{kj}{}_i + \ldots$$

cf. electrodynamics: $p_i = \hat{p}_i - qA_i$

- Canonical field momentum $\hat{\pi}^{ijTT}$ has delta-corrections:

$$\pi^{ijTT} = \hat{\pi}^{ijTT} + \frac{4\pi G}{m^2}\hat{p}_m\hat{p}_k\hat{S}^{lm}\delta^{TTij}_{kl}\delta + \ldots$$

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PN Counting with Spin

for Hamiltonians

for maximally rotating objects: $S = \frac{Gm^2\chi}{c}$ $\chi = 1$

order		1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
	H^{N}								
PM		$+ H^{1\mathrm{PN}}$		$+ H^{2\mathrm{PN}}$	$+ H^{2.5\mathrm{PN}}$	$+ H^{3\mathrm{PN}}$	$+ H^{3.5\mathrm{PN}}$	$+ H^{4\mathrm{PN}}$	$+ H^{4.5\mathrm{PN}}$
SO			$+ H^{\mathrm{LO}}_{\mathrm{SO}}$		$+ H^{\mathrm{NLO}}_{\mathrm{SO}}$		$+ H^{\mathrm{N^2LO}}_{\mathrm{SO}}$	$+ H^{\mathrm{LO,R}}_{\mathrm{SO}}$	$+ H^{\mathrm{N^3LO}}_{\mathrm{SO}}$
S_1^2				$+ H^{\mathrm{LO}}_{S_1^2}$		$+ H^{\mathrm{NLO}}_{S_1^2}$		$+ H^{\mathrm{N^2LO}}_{S_1^2}$	$+ H^{\mathrm{LO,R}}_{S_1^2}$
S_1S_2				$+ H^{\mathrm{LO}}_{S_1S_2}$		$+ H^{\mathrm{NLO}}_{S_1S_2}$		$+ H^{\mathrm{N^2LO}}_{S_1S_2}$	$+ H^{\mathrm{LO,R}}_{S_1S_2}$
spin^3							$+ H^{\mathrm{LO}}_{S^3}$		$+ H^{\mathrm{NLO}}_{S^3}$
spin^4								$+ H^{\mathrm{LO}}_{S^4}$	
$\vdots$									$\ddots$

H known EOM known for Black Holes not known (yet)

Radiation field known to 2.5PN order, multipoles to 3PN order.

Results for Spin Hamiltonians

shown for equal masses, circular orbits, and aligned spins

$$H_{\text{spin}} = H_{S_1 O} + H_{S_2 O} + H_{S_1^2} + H_{S_2^2} + H_{S_1 S_2} + H_{S^3} + H_{S^4} + \ldots$$

LO NLO N^{2}LO

$$H_{S_1 O} = S_1 L \left\{ \frac{7}{8r^3} + \frac{3}{r^4}\left[-1 + \frac{5}{16}\frac{L^2}{r}\right] + \frac{1}{64r^5}\left[401 - \frac{751}{8}\frac{L^2}{r} - \frac{25}{16}\frac{L^4}{r^2}\right] + \ldots \right\}$$

$$H_{S_1^2} = S_1^2 \left\{ -\frac{C_{ES^2}}{8r^3} + \frac{1}{16r^4}\left[6C_{ES^2} + 5 - \frac{17C_{ES^2} - 11}{4}\frac{L^2}{r}\right] + \ldots \right\}$$

$$H_{S_1 S_2} = S_1 S_2 \left\{ -\frac{1}{4r^3} + \frac{1}{2r^4}\left[3 - \frac{7}{8}\frac{L^2}{r}\right] + \frac{1}{64r^5}\left[-271 - 238\frac{L^2}{r} + \frac{45}{8}\frac{L^4}{r^2}\right] + \ldots \right\}$$

$$H_{S^3} = \frac{5L}{64r^5}(S_1 + S_2)^3 + \ldots$$

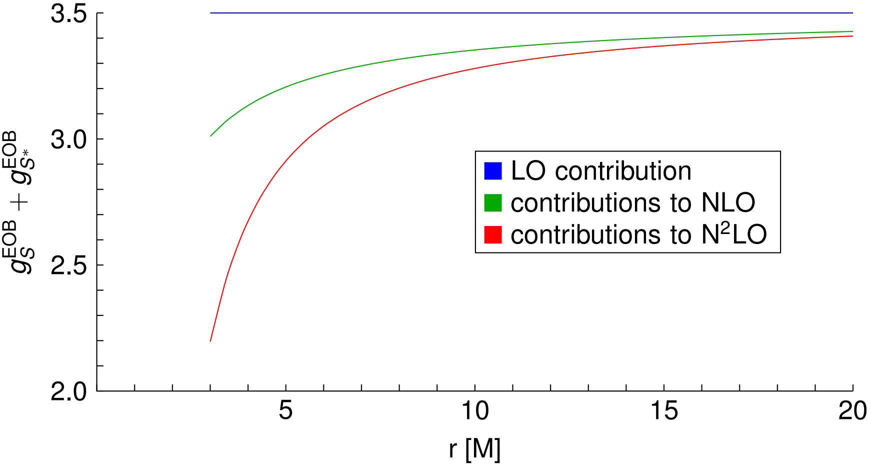
yet only known

$$H_{S^4} = -\frac{3}{128r^5}(S_1 + S_2)^4 + \ldots$$

for black holes

Spin-Orbit: Gyro-Gravitomagnetic Ratios $g_S^{EOB} + g_{S^*}^{EOB}$

for equal masses and circular orbits, A. Nagar arXiv:1106.4349



Conclusions & Outlook

Conclusions:

- Spin (linear order): universal
- Quadrupole: internal structure, EOS
- Effects are small, but
  - accumulate during inspiral
  - become increasingly important in late inspiral
- Parameter space considerably increased!

Possible Future Tasks:

- Spin part of radiation field at 3PN (and beyond)
- Spin Hamiltonians:
  - $H^{\mathrm{LO}}_{S^3}$ and $H^{\mathrm{LO}}_{S^4}$ for (neutron) stars
  - $H^{\mathrm{N^2LO}}_{S_1^2}$ at 4PN
- More on tidal deformations
- Dynamical quadrupole $L_Q \sim E_{\mu\nu} Q^{\mu\nu}$

Thank you for your attention

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