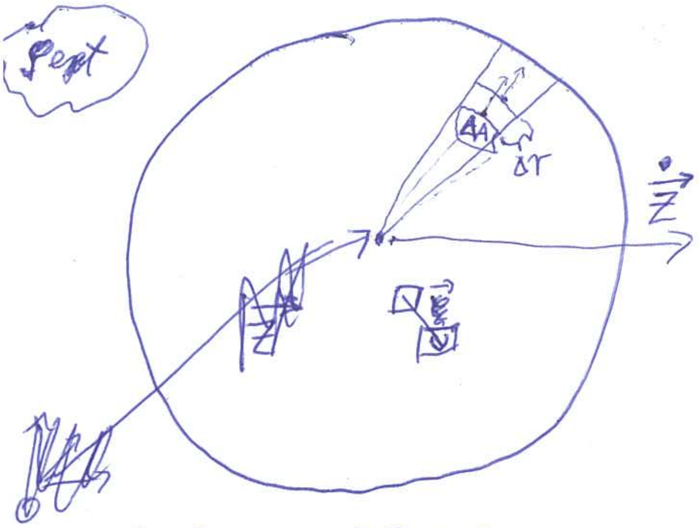
Oscillation modes, tides, and resonances of compact stars

arxiv.org/abs/1306.5820
1304.2228



neutron star
↳ temperature negligible
↳ barotropic equation of state $P = P(\rho)$

P ↓ pressure, ρ ↓ massdensity

star in ~~equilibrium (subscript "0")~~

gravitational potential Φ

star in equilibrium
(subscript "0")

nonrotating, spherically symmetric

net force on mass element: $\cancel{P' = \frac{dP}{dr}}$ $P_0' \cdot \Delta r \cdot \Delta A$ $\quad ' = \frac{d}{dr}$

compensated by gravity: $-\Phi_0' \cdot \Delta m = -\Phi_0' \rho_0 \Delta r \Delta A$

↳ $P_0' = -\rho_0 \Phi_0'$ (hydrostatic equilibrium)

gravity: $\Delta\Phi_0 = \frac{1}{r^2}\frac{d(r^2\Phi_0')}{dr} = 4\pi G \rho_0$

linear perturbation
(subscript "1")

physical displacement $\vec{\xi}$ of mass elements

action: $S_{full} = \int dt\, L_{full}$

split perturbation Langrangian

	unperturbed	perturbed
velocity $\vec{u}$	$\vec{u}_0 = 0$	$\vec{u}_1 = \dot{\vec{\xi}}$
massdensity ρ	$\rho_0 = \rho_0(r)$	$\rho_1 = -\nabla\cdot(\vec{\xi}\rho_0)$ (mass conservation)
pressure P	$P_0 = P_0(r)$	$P_1 = c_s^2 \rho_1$
ext. source ρ_{ext}	$\rho_{ext\,0} = 0$	$\rho_{ext\,1} = \rho_{ext}$
grav. field Φ	$\Phi_0 = \Phi_0(r)$	Φ_1

adiabatic speed of sound c_s: $c_s^2 = \frac{dP_0}{d\rho_0}$

$\dot{\rho} + \nabla\cdot(\rho\vec{u}) = 0$

$$L_{full} = L_{\Phi_1} + L_{kin} + L_{star} + L_{coupl}$$

grav. pert.: $L_{\Phi_1} = \int d^3x \frac{1}{8\pi G}\Phi_1 \Delta\Phi_1$

kin. Energy: $L_{kin} = \frac{1}{2} m \dot{\vec{Z}}^2$

coupling to ext. source ρ_{ext}: $L_{coupl.} = -\int d^3x\, \rho_{ext}(\Phi_0 + \Phi_1)$

notice: $\Phi = \Phi_0 + \Phi_1 = -\frac{\delta L_{full}}{\delta \rho_{ext}}$

star interior:

$$L_{star} = \int d^3x \left[\underbrace{\tfrac{1}{2}\rho_0 \dot{\vec{\xi}}^2_{COM}}_{\text{in center-of-mass system (noninertial)}} - \underbrace{(\rho E)_2}_{\text{2nd pert. of interal energy}} - \rho_0 \vec{\xi} \cdot (\nabla \phi_1 + \underbrace{\ddot{\vec{Z}}}_{\text{fictitious force due to acceleration}}) \right]$$

2nd pert. of interal energy

E: specific " "

all vibration energies added up!

$$(\rho E)_2 \overset{E=E(\rho)}{=} \rho_1 E_1 + \rho_0 E_2 + \cancel{\rho_2 E_0}$$

relevant for 2nd perturb.

$$= \rho_1 \frac{dE_0}{d\rho_0}\rho_1 + \rho_0 \frac{1}{2}\frac{d^2E_0}{d\rho_0^2}\rho_1^2$$

1st law of thermodyn.: $dE = -P\, d\rho^{-1} = +\frac{P}{\rho^2} d\rho$

$$\frac{dE}{d\rho} = \frac{P}{\rho^2}, \quad \frac{d^2E}{d\rho} = -2\frac{P}{\rho^3} + \frac{1}{\rho^2}\frac{dP}{d\rho}$$

$$= \cancel{\frac{P_0}{\rho_0^2}\rho_1^2} - \cancel{\frac{P_0}{\rho_0^2}\rho_1^2} + \frac{1}{2\rho_0}\underbrace{\frac{dP_0}{d\rho_0}}_{c_s^2}\rho_1^2$$

$$\boxed{(\rho E)_0 = \frac{c_s^2}{2\rho_0}\rho_1^2 = \frac{c_s^2}{2\rho_0}\left[\nabla \cdot (\rho_0 \vec{\xi})\right]^2}$$

Normal modes

elimination of the grav. field pert. ϕ_1

$\delta\phi_1$-variation

$$\hookrightarrow \frac{1}{4\pi G}\Delta\phi_1 - \rho_{ext} + \underbrace{\nabla\cdot(\rho_0\vec{\xi})}_{-\rho_1} = 0$$

$$\Rightarrow \phi_1 = 4\pi G\,\Delta^{-1}[\rho_{ext} - \underbrace{\cancel{\nabla\cdot(\rho_0\vec{\xi})}}_{\rho_1}] \equiv$$

insert into action:

$$L_{\phi_1} = \int d^3x \frac{1}{8\pi G} 4\pi G\,\Delta^{-1}[\rho_{ext} \overset{+}{-} \overset{\rho_1}{\cancel{\nabla\cdot(\rho_0\vec{\xi})}}]\,\underbrace{\overset{4\pi G}{\Delta\Delta^{-1}}}_{1}[\rho_{ext} \overset{+}{-} \overset{\rho_1}{\cancel{\nabla\cdot(\rho_0\vec{\xi})}}]$$

$$= \int d^3x \frac{1}{2}4\pi G\left(\rho_{ext}\Delta^{-1}\rho_{ext} \pm \rho_{ext}\Delta^{-1}\overset{\rho_1}{\cancel{\nabla\cdot(\rho_0\vec{\xi})}} \overset{+}{-} \Delta^{-1}\rho_{ext}\overset{\rho_1}{\cancel{\nabla\cdot(\rho_0\vec{\xi})}} + \overset{\rho_1}{\cancel{\nabla\cdot(\rho_0\vec{\xi})}}\Delta^{-1}\overset{\rho_1}{\cancel{\nabla\cdot(\rho_0\vec{\xi})}}\right)$$

Δ Hermitian

$$L_{couple} = -\int d^3x\,\rho_{ext}(\underset{4\pi G\Delta^{-1}\rho_0}{\phi_0} + 4\pi G\,\Delta^{-1}[\rho_{ext} \cancel{-\nabla\cdot(\rho_0\vec{\xi})} + \rho_1])$$

$$= -\int d^3x\, 4\pi G\,\Delta^{-1}\rho_{ext}\,(\rho_0 + \rho_1 + \rho_{ext})$$

$$L_{star} = \int d^3x\left(\ldots + \rho_0\vec{\xi}\cdot\nabla\phi_1\right) = \int d^3x\left(\ldots \mp \rho_1\phi_1\right)$$

factor -1

$$= \int d^3x\left[\ldots + \rho_1 4\pi G\,\Delta^{-1}(\rho_{ext} + \rho_1)\right]$$

collect all terms quadratic in $\vec{\xi}$ (or ρ_1):

$$L_{NM} = \int d^3x\left[\frac{\rho_0}{2}\dot{\vec{\xi}}^2_{com} \mp \frac{1}{2}4\pi G\,\rho_1\Delta^{-1}\rho_1 - \frac{c_s^2}{2\rho_0}\rho_1^2\right]$$

$$\rho_1 = -\nabla\cdot(\rho_0\vec{\xi}) \qquad \frac{1}{2}\rho_0\vec{\xi}\cdot$$

$$= \int d^3x\left[\frac{\rho_0}{2}\dot{\vec{\xi}}^2_{com} - \underbrace{\nabla\left(-\frac{c_s^2}{\rho_0}\cancel{\nabla} - 4\pi G\,\Delta^{-1}\right)\nabla\cdot(\rho_0\vec{\xi})}_{\mathcal{D}\vec{\xi}}\right]$$

$$\hookrightarrow \frac{1}{2}\rho_0\vec{\xi}\,\mathcal{D}\vec{\xi}$$

$\mathcal{D}$ is:
- linear operator
- nonlocal
- Hermitian w.r.t. $dm_0 = \rho_0 d^3x$!

collect terms quadratic in ρ_{ext}:

$$L_{ext} = -\int d^3x \, \frac{1}{2} \rho_{ext} \underbrace{4\pi G \Delta^{-1} \rho_{ext}}_{:= \phi_{ext}}$$

collect terms linear in ρ_{ext} and $\ddot{\vec{Z}}$:

$$L_{int} = -\int d^3x \left[\phi_{ext} (\rho_0 + \rho_1) + \underbrace{\rho_0 \vec{\xi}_h \cdot \ddot{\vec{Z}}}_{\rho_0 \vec{\xi}_h \nabla \cdot (\vec{X} \cdot \ddot{\vec{Z}}) \sim \rho_1 \vec{X} \cdot \ddot{\vec{Z}}} \right]$$

Now: $L_{full} = L_{kin} + L_{NM} + L_{int} + L_{ext}$

$\mathcal{D}$ Hermitian $\Rightarrow$ $\mathcal{D} \, \underbrace{\vec{\xi}^{NM}_{n\ell m}} = \omega^2_{n\ell} \, \vec{\xi}^{NM}_{n\ell m}$ $\boxed{\text{normal modes!}}$

rot. symmetry

$\omega^2_{n\ell} \in \mathbb{R}$

angular momentum "quantum" numbers

Normalisation: $\int d^3x \, \rho_0 \, \vec{\xi}^{NM\dagger}_{n'\ell'm'} \vec{\xi}^{NM}_{n\ell m} = \delta_{n'n} \delta_{\ell'\ell} \delta_{m'm}$

amplitude formulation

$\vec{\xi}^{NM}_{n\ell m}$ complete

$\hookrightarrow \vec{\xi} = \sum_{n\ell m} A_{n\ell m}(t)\, \vec{\xi}^{NM}_{n\ell m}(\vec{x})$

insert into Lagrangian:

$$L_{NM} = \int d^3x \, \frac{\rho_0}{2}\left[\dot{\vec{\xi}}^* \cdot \dot{\vec{\xi}} - \vec{\xi}^* \mathcal{D} \vec{\xi}\right]$$

$$\cancel{= \int d^3x \frac{\rho_0}{2}[A}$$

$$= \sum_{n\ell m} \frac{1}{2}\left[|\dot{A}_{n\ell m}|^2 - \omega_{n\ell}^2 |A_{n\ell m}|^2\right]$$

$$L_{int} = -\int d^3x \left[\cancel{\phi_{ext}\rho_0 - \nabla\cdot(\rho_0\vec{\xi})}\, \phi_{ext}\rho_0 - \nabla\cdot(\rho_0 \vec{\xi})(\phi_{ext} + \vec{x}\cdot\ddot{\vec{Z}})\right]$$

$$= -\int d^3x\, \phi_{ext}\rho_0 + \sum_{n\ell m} A_{n\ell m} \underbrace{\left[\int d^3x \underbrace{\nabla\cdot(\rho_0 \vec{\xi}^{NM}_{n\ell m})}_{-\rho^{NM}_{n\ell m} \sim Y^{\ell m}} (\phi_{ext} + \vec{x}\cdot\ddot{\vec{Z}})\right]}_{=: f^*_{n\ell m}}$$

equations of motion:

$$\ddot{A}_{n\ell m} + \omega_{n\ell}^2 A_{n\ell m} = f_{n\ell m}$$

harmonic oscillator!

notice: $\vec{\xi} = \vec{\xi}^* \Rightarrow A^*_{n\ell m} = (-1)^m A_{n\ell -m}$

Overlap integrals

decompose $\rho^{NM}_{n\ell m} = \rho_{n\ell}(r)\, Y^{\ell m}(\theta,\varphi)$; $\phi_{ext} = \sum_{\ell m} \underbrace{\phi^{\ell m}_{ext}}_{1/\ell! \text{ const.}} r^\ell Y^{\ell m}$ (see paper)

$f_{n\ell m} = -\int d^3x\, \rho^{NM*}_{n\ell m}\, \phi_{ext}$ for $\ell \geq 2$ ($\vec{x} \sim Y^{1m}$)

$$\cancel{= -\sum_{\ell m} \phi^{\ell m}_{ext} \int dr\, r^{2+\ell} \rho^{NM}_{n\ell}(r)}$$

$$= -\frac{1}{\ell!}\phi^{\ell m}_{ext} \underbrace{\int dr\, r^{2+\ell} \rho^{NM}_{n\ell}}_{I_{n\ell} = \text{overlap integral}}$$

multipoles:

$$q_{\ell m} \equiv \int d^3x\, \underbrace{\rho_1}_{\sum_{n\ell' m'} A_{n\ell' m'} \rho^{NM}_{n\ell'}(r) Y^{\ell' m'}} r^\ell Y^{\ell m *}$$

$$= \sum_n A_{n\ell m} \int dr\, r^{2+\ell} \rho^{NM}_{n\ell} = \sum_n A_{n\ell m} I_{n\ell}$$

Fourier domain "~":

$$-\omega^2 \tilde{A}_{n\ell m} + \omega_{n\ell}^2 \tilde{A}_{n\ell m} = -\frac{1}{\ell!} I_{n\ell}\, \tilde{\phi}^{\ell m}_{ext}$$

$$\tilde{q}_{\ell m} = \sum_n \tilde{A}_{n\ell m} I_{n\ell}$$

$$= -\frac{1}{\ell!} \underbrace{\sum_n \frac{I_{n\ell}^2}{\omega_{n\ell}^2 - \omega^2}}_{\tilde{F}_\ell = \text{linear response of } \ell\text{-pole to ext. field}} \tilde{\phi}^{\ell m}_{ext}$$