The PN Approximation beyond Point-Masses

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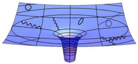

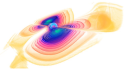

DFG : SFB/TR7, GRK 1523, and STE 2017/1-1
FCT : PTDC/CTEAST/098034/2008

The PN Approximation beyond Point-Masses

$$\sqrt{-g}T^{\mu\nu}(x^\sigma) = \int d\tau \Big[u^{(\mu} p^{\nu)} \delta_{(4)} + \left(u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha}$$

$$+ \frac{1}{3} R_{\alpha\beta\rho}{}^{(\mu} J^{\nu)\rho\beta\alpha} \delta_{(4)} - \frac{2}{3} \left(J^{\mu\alpha\beta\nu} \delta_{(4)} \right)_{||(\alpha\beta)} + \dots \Big]$$

$$u^\mu = \frac{dz^\mu}{d\tau} \qquad \delta_{(4)} = \delta(z^\sigma - x^\sigma)$$

- Point masses only distinguished by a mass $m = \sqrt{-p_\mu p^\mu}$
- Adding a dipole: Spin
- Higher multipoles: Quadrupole, octupole, ... ("finite size effects")
- Problem for GW astronomy: Parameter space becomes very big

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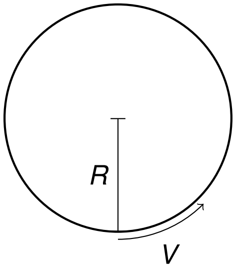
Two Facts on Spin in Relativity

1. Minimal Extension



- ring of radius R and mass M
- spin: $S = R\,M\,V$
- maximal velocity: $V \leq c$
  $\Rightarrow$ minimal extension:

$$R = \frac{S}{MV} \geq \frac{S}{Mc}$$

2. Center-of-mass

- now moving with velocity v
- relativistic mass changes inhom.
- frame-dependent center-of-mass
- need spin supplementary condition,

$$\text{e.g.,} \quad S^{\mu\nu} p_\nu = 0$$

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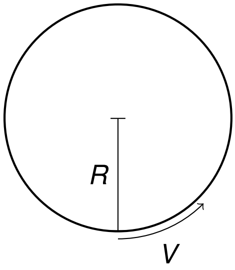

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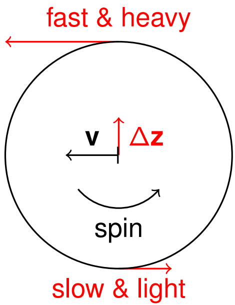

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Modeling of Quadrupole Deformation via an Action

$$L = m_c \underbrace{\sqrt{-u_\mu u^\mu}}_{u} + \underbrace{\frac{1}{m_c u} B_{\mu\nu} S^\mu u_\alpha S^{\alpha\nu}}_{\text{SSC preserving}} + \underbrace{\frac{C_{ES^2}}{2 m_c u} E_{\mu\nu} S^\mu{}_\alpha S^{\alpha\nu}}_{\text{deformation due to spin}} + \underbrace{\frac{\mu_2}{4u^3} E_{\mu\nu} E^{\mu\nu}}_{\text{tidal deformation}} + ...$$

$$E_{\mu\nu} \sim R_{\mu\alpha\nu\beta} u^\alpha u^\beta \qquad B_{\mu\nu} \sim \tfrac{1}{2} \epsilon_{\mu\rho\alpha\beta} R_{\nu\sigma}{}^{\alpha\beta} u^\rho u^\sigma \qquad S^\mu = \tfrac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu S_{\alpha\beta}$$

- m_c, C_{ES^2}, and μ_2 are assumed to be constants
- Notice: $m_c \neq m$
- From Bailey, Israel (1975):

$$J^{\mu\nu\alpha\beta} = -6 \frac{\partial L}{\partial R_{\mu\nu\alpha\beta}}$$

- $J^{\mu\nu\alpha\beta}$ contains mass, flow, and stress quadrupoles
- Covariant mass quadrupole: (for $u = 1$)

$$\text{mass quadrupole} \sim 2 \frac{\partial L}{\partial E_{\mu\nu}} = \frac{C_{ES^2}}{m_c} S^\mu{}_\alpha S^{\alpha\nu} + \mu_2 E^{\mu\nu}$$

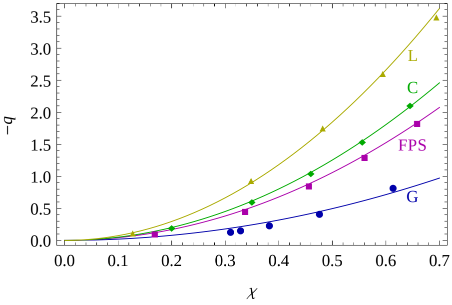
Quadrupole Deformation due to Spin

Matching the Coefficient C_{ES^2} for Neutron Stars, Laarakkers, Poisson gr-qc/9709033

- Here $m = 1.4 M_\odot$
- Dim.-less mass quadrupole: q
- Dim.-less spin: χ
- Quadratic fit is extremely good:

$$-q \approx C_{ES^2} \chi^2$$

- $C_{ES^2} = 4.3 \ldots 7.4$, EOS dependent
- Also depends on mass



see Laarakkers, Poisson gr-qc/9709033

- A single star is enough for the matching
- S^4-coupling $E_{\mu\nu} S^{\mu}{}_{\rho} S^{\rho\nu} S^2$ is highly suppressed
- For black holes $C_{ES^2} = 1$

PN Counting with Spin

for maximally rotating objects: $S = \frac{Gm^2\chi}{c}$ $\chi = 1$

order	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
H^{N}								
PM	$+H^{1\mathrm{PN}}$		$+H^{2\mathrm{PN}}$	$+H^{2.5\mathrm{PN}}$	$+H^{3\mathrm{PN}}$	$+H^{3.5\mathrm{PN}}$	$+H^{4\mathrm{PN}}$	$+H^{4.5\mathrm{PN}}$
SO		$+H_{\mathrm{SO}}^{\mathrm{LO}}$		$+H_{\mathrm{SO}}^{\mathrm{NLO}}$		$+H_{\mathrm{SO}}^{\mathrm{N^2LO}}$	$+H_{\mathrm{SO}}^{\mathrm{LO,R}}$	$+H_{\mathrm{SO}}^{\mathrm{N^3LO}}$
S_1^2			$+H_{S_1^2}^{\mathrm{LO}}$		$+H_{S_1^2}^{\mathrm{NLO}}$		$+H_{S_1^2}^{\mathrm{N^2LO}}$	$+H_{S_1^2}^{\mathrm{LO,R}}$
S_1S_2			$+H_{S_1S_2}^{\mathrm{LO}}$		$+H_{S_1S_2}^{\mathrm{NLO}}$		$+H_{S_1S_2}^{\mathrm{N^2LO}}$	$+H_{S_1S_2}^{\mathrm{LO,R}}$
spin^3						$+H_{S^3}^{\mathrm{LO}}$		$+H_{S^3}^{\mathrm{NLO}}$
spin^4							$+H_{S^4}^{\mathrm{LO}}$	
⋮								⋱

H known EOM known for Black Holes not known (yet)

Radiation field known to 2.5PN order, multipoles to 3PN order.

Results for Hamiltonians

shown for equal masses, circular orbits, and aligned spins

$$H = H_{\mathrm{PM}} + H_{\mathrm{S_1 O}} + H_{\mathrm{S_2 O}} + H_{\mathrm{S_1^2}} + H_{\mathrm{S_2^2}} + H_{\mathrm{S_1 S_2}} + H_{\mathrm{S^3}} + H_{\mathrm{S^4}} + \cdots + H_{\mathrm{tidal}} + \ldots$$

LO NLO N^2LO

$$H_{\mathrm{S_1 O}} = S_1 L \left\{ \frac{7}{8r^3} + \frac{3}{r^4}\left[-1 + \frac{5}{16}\frac{L^2}{r}\right] + \frac{1}{64r^5}\left[401 - \frac{751}{8}\frac{L^2}{r} - \frac{25}{16}\frac{L^4}{r^2}\right] + \ldots \right\}$$

$$H_{\mathrm{S_1^2}} = S_1^2 \left\{ -\frac{C_{ES^2}}{8r^3} + \frac{1}{16r^4}\left[6C_{ES^2} + 5 - \frac{17C_{ES^2} - 11}{4}\frac{L^2}{r}\right] + \ldots \right\}$$

$$H_{\mathrm{S_1 S_2}} = S_1 S_2 \left\{ -\frac{1}{4r^3} + \frac{1}{2r^4}\left[3 - \frac{7}{8}\frac{L^2}{r}\right] + \frac{1}{64r^5}\left[-271 - 238\frac{L^2}{r} + \frac{45}{8}\frac{L^4}{r^2}\right] + \ldots \right\}$$

$$H_{\mathrm{S^3}} = \frac{5L}{64r^5}(S_1 + S_2)^3 + \ldots$$

yet only known

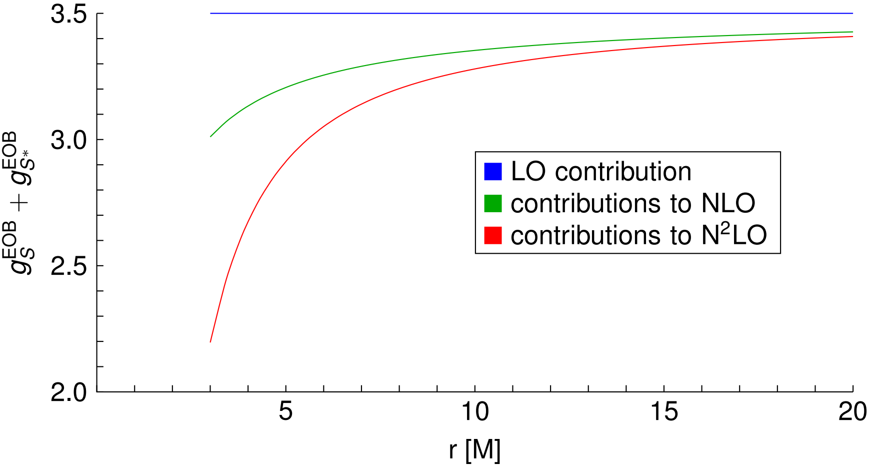
$$H_{\mathrm{S^4}} = -\frac{3}{128r^5}(S_1 + S_2)^4 + \ldots$$

for black holes

H_{tidal}: LO/EOB in Damour, Nagar, arXiv:0911.5041
NLO mass-quadrupole in Vines, Flanagan, arXiv:1009.4919

Spin-Orbit: Gyro-Gravitomagnetic Ratios $g_S^{\mathrm{EOB}} + g_{S^*}^{\mathrm{EOB}}$

for equal masses and circular orbits, A. Nagar arXiv:1106.4349



Conclusions

Future Tasks:

- Calculate spin part of radiation field at 3PN (and beyond)
- Calculation of spin Hamiltonians:
  - $H^{\mathrm{LO}}_{S^3}$ and $H^{\mathrm{LO}}_{S^4}$ for (neutron) stars
  - $H^{\mathrm{N^2LO}}_{\mathrm{S}_1^2}$ at 4PN
  - $H^{\mathrm{N^3LO}}_{\mathrm{SO}}$ and $H^{\mathrm{NLO}}_{S^3}$ at 4.5PN (later)
- More on tidal deformations

Questions:

- What are the most interesting future subjects for PN theory?
- What is most relevant for GW astronomy?
- How to cover parameter space with spin for Advanced LIGO (2014)?

Thank you for your attention

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