Exercises

1. Verify that $\mathbf{A}_{(2)}$, i.e., the solution of

$$\Delta\mathbf{A}_{(2)} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{A}_{(2)}) = -4\pi G\mathbf{j}_{(0)}, \quad \mathbf{j}_{(0)} = \mathbf{p}_1\delta_1 - \frac{1}{2}\mathbf{S}_1\times\nabla\delta_1 + (1\leftrightarrow 2)\,, \tag{1}$$

is given by

$$\mathbf{A}_{(2)} = \frac{G\mathbf{p}_1}{r_1} - \frac{G}{8}(\mathbf{p}_1\cdot\nabla)\nabla r_1 - \frac{G}{2}\mathbf{S}_1\times\nabla\left(\frac{1}{r_1}\right) + (1\leftrightarrow 2)\,. \tag{2}$$

Here $\delta_a = \delta(\mathbf{x}-\mathbf{q}_a)$ and $r_a = |\mathbf{x}-\mathbf{q}_a|$. (Hint: Use $\Delta r_1 = 2/r_1$ and $\Delta(1/r_1) = -4\pi\delta_1$ whenever possible.)

2. Verify that the expression

$$H_{\mathrm{1PN}} = \frac{1}{c^2}\int \mathrm{d}^3x\,\left(\rho_{(4)} - \frac{1}{2}\rho_{(2)}\varphi_{(0)} - \frac{1}{2}\rho_{(0)}\varphi_{(2)} + 2\,\mathbf{j}_{(0)}\cdot\mathbf{A}_{(2)} + \frac{1}{4}\rho_{(0)}\varphi_{(0)}^2\right) \tag{3}$$

simplifies to

$$H_{\mathrm{1PN}} = \frac{1}{c^2}\int \mathrm{d}^3x\,\left(\rho_{(4)} - \rho_{(2)}\varphi_{(0)} + 2\,\mathbf{j}_{(0)}\cdot\mathbf{A}_{(2)} + \frac{1}{2}\rho_{(0)}\varphi_{(0)}^2\right) \tag{4}$$

after partial integration of $\rho_{(0)}\varphi_{(2)} = -\frac{1}{4\pi G}(\Delta\varphi_{(0)})\varphi_{(2)}$ and use of $\Delta\varphi_{(2)} = -4\pi G(\rho_{(2)} - \frac{1}{2}\rho_{(0)}\varphi_{(0)})$.

3. Calculate the following Hamiltonians by performing the integration in equation (4):

   a) The leading order (LO) $\mathrm{S}_1\mathrm{S}_2$ Hamiltonian

$$H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{S}_2} = \frac{G}{c^2 r_{12}^3}\left(3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12}) - (\mathbf{S}_1\cdot\mathbf{S}_2)\right) \tag{5}$$

   as the part of equation (4) quadratic in spin. Here $r_{12} = |\mathbf{q}_1-\mathbf{q}_2|$ and $\mathbf{n}_{12} = (\mathbf{q}_1-\mathbf{q}_2)/r_{12}$.

   b) The leading order spin-orbit (SO) Hamiltonian

$$H^{\mathrm{LO}}_{\mathrm{SO}} = \frac{G}{c^2 r_{12}^2}(\mathbf{S}_1\times\mathbf{n}_{12})\cdot\left(\frac{3m_2}{2m_1}\mathbf{p}_1 - 2\mathbf{p}_2\right) + (1\leftrightarrow 2) \tag{6}$$

   as the part of equation (4) linear in spin.

   c) The 1PN point-mass (PM) Hamiltonian

$$H^{\mathrm{PM}}_{\mathrm{1PN}} = -\frac{(\mathbf{p}_1^2)^2}{8c^2m_1^3} - \frac{(\mathbf{p}_2^2)^2}{8c^2m_2^3} + \frac{G}{c^2 r_{12}}\bigg(-\frac{3m_2}{2m_1}\mathbf{p}_1^2 - \frac{3m_1}{2m_2}\mathbf{p}_2^2 + \frac{7}{2}(\mathbf{p}_1\cdot\mathbf{p}_2) + \frac{1}{2}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})\bigg) + \frac{G^2}{2c^2r_{12}^2}(m_1^2m_2 + m_2^2m_1) \tag{7}$$

   as the spin-independent part of equation (4).

Use the formulas for $\mathbf{A}_{(2)}$ and $\mathbf{j}_{(0)}$ given in exercise 1 as well as

$$\rho_{(0)} = m_1\delta_1 + m_2\delta_2 \qquad \varphi_{(0)} = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2} \tag{8}$$

$$\rho_{(2)} = \frac{\mathbf{p}_1^2}{2m_1}\delta_1 - \frac{1}{2m_1}\mathbf{p}_1\cdot(\mathbf{S}_1\times\nabla\delta_1) + (1\leftrightarrow 2) \tag{9}$$

$$\rho_{(4)} = -\frac{(\mathbf{p}_1^2)^2}{8m_1^3}\delta_1 - \frac{\mathbf{p}_1^2}{m_1}\varphi_{(0)}\delta_1 + \frac{1}{m_1}\varphi_{(0)}\mathbf{p}_1\cdot(\mathbf{S}_1\times\nabla\delta_1) + (1\leftrightarrow 2) \tag{10}$$

Drop divergent integrals of the form $\int \mathrm{d}^3x\,\delta_1/r_1$ (undefined self-interactions).

Solution of 3.a

The only term inside the integral in equation (4) contributing to the quadratic-in-spin level is $2\,\mathbf{j}_{(0)} \cdot \mathbf{A}_{(2)}$. All integrals quadratic in $\mathbf{S}_1$ or $\mathbf{S}_2$ are divergent and can be dropped. Finally one ends up with

$$\frac{G}{2c^2}\int \mathrm{d}^3x \left[\mathbf{S}_2 \times \nabla\left(\frac{1}{r_2}\right)\right] \cdot (\mathbf{S}_1 \times \nabla\delta_1) \tag{11}$$

$$= \frac{G}{2c^2}\int \mathrm{d}^3x \left[\mathbf{S}_2 \times \left(-\frac{\mathbf{n}_2}{r_2^2}\right)\right] \cdot (\mathbf{S}_1 \times \nabla\delta_1) \tag{12}$$

$$= \frac{G}{2c^2}\int \mathrm{d}^3x \, \frac{1}{r_2^2}[(\mathbf{S}_1 \cdot \mathbf{n}_2)(\mathbf{S}_2 \cdot \nabla\delta_1) - (\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{n}_2 \cdot \nabla\delta_1)] \tag{13}$$

$$= -\,(\mathbf{S}_2 \cdot \nabla)\frac{G}{2c^2 r_2^2}(\mathbf{S}_1 \cdot \mathbf{n}_2)\bigg|_{\mathbf{x}=\mathbf{q}_1} + (\mathbf{S}_1 \cdot \mathbf{S}_2)\left[\nabla \cdot \left(\frac{G\mathbf{n}_2}{2c^2 r_2^2}\right)\right]\bigg|_{\mathbf{x}=\mathbf{q}_1} \tag{14}$$

$$= -\,\frac{G}{2c^2 r_2^3}((\mathbf{S}_2 \cdot \mathbf{S}_1) - 3(\mathbf{S}_2 \cdot \mathbf{n}_2)(\mathbf{S}_1 \cdot \mathbf{n}_2))\bigg|_{\mathbf{x}=\mathbf{q}_1} + 0 \tag{15}$$

$$= \frac{G}{2c^2 r_{12}^3}(3(\mathbf{S}_2 \cdot \mathbf{n}_{12})(\mathbf{S}_1 \cdot \mathbf{n}_{12}) - (\mathbf{S}_2 \cdot \mathbf{S}_1)) \tag{16}$$

and another integral with particle labels 1 and 2 exchanged (this just gives an overall factor of 2). Here $\mathbf{n}_a = (\mathbf{x} - \mathbf{q}_a)/r_a$.