Newtonian Hamiltonian and Spin Effects in General Relativity (GR)

Outline

I. ~~the~~ post-Newtonian (PN) approximation

II. ~~the~~ 1PN ADM Hamiltonian (similar Edyn)
(ADM ≙ Arnowitt, Deser, Misner)

III. point-masses and dipoles (spin)

IV. gravitational spin-orbit interaction

I. ~~the post-Ne~~ PN approximation

Newtonian Limit: - weak field ~ G small
- slow motion ~ $\frac{v}{c}$ small

- limit $\frac{1}{c} \to 0$ implies ~~small G and~~ v and G small

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu},$$

(also c's from $T^{\mu\nu}$)

~~and indeed this~~ leads to Newtonian limit

to ~~what~~ which ~~kind of system~~ situation ~~is best describe~~ an expansion in $\frac{1}{c}$ fits ~~the~~ best?

(one could expand in G and v indep.)

↳ Virial Theorem in Newtonian Gravity for a binary System:

$(2\langle T\rangle_t = -\langle V\rangle_t)$

$$\frac{v^2}{c^2} \sim \frac{GM}{c^2 r}$$

(≙ dimensionless expansion parameter $\ll 1$)

one ~~PN order ≙ c^{-2}~~

(relative order!)
(there may be half orders)

PN orders of the metric components follow from the source terms:

$T^{00} \sim \rho c^2$, $T^{0i} \sim c j^i$, $T^{ij} = \mathcal{O}(c^0)$

↑ massdensity ↑ masscurrent density

first at $\cancel{T^{ij} = \mathcal{O}(c^0)}$ N(ewton) 1PN 2PN

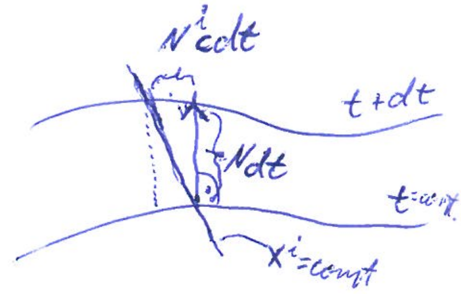
- valid for bound system: inspiralling binary

- ~~makes statements for~~ time averaged

one PN order ≙ c^{-2}

II. 1PN ADM Hamiltonian

3+1 split:



$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$$
$$= -\underset{\substack{\uparrow\\ \text{lapse}}}{N^2}c^2 dt^2 + g_{ij}(dx^i + \underset{\substack{\uparrow\\ \text{shift}}}{N^i} c\,dt)(dx^j + N^j c\,dt)$$

unit normal ~~vector~~ to spatial hypersurfaces:

$$n_\mu = (-N, \vec{0}), \quad n_\mu n^\mu = 1$$

~~to 1PN accuracy it holds:~~

to 1PN accuracy one can ~~set~~ choose (gauge choice)

$$g_{ij} = \left(1 + \frac{\varphi}{2c^2}\right)^4 \delta_{ij}$$

[Schwarzschild metric in isotropic coordinates: $\varphi = \frac{Gm}{r}$]

1PN field equations (constraints), ~~with $N^i c = -4A^i$~~

$$\Delta\varphi = \varphi_{,ii} = -4\pi G\,\varrho_{eff}$$

(orders of metric comp. follows...)

$$+\frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) + \Delta\vec{A} = \frac{4\pi G}{c^2}\vec{j}$$

with $c^2\underbrace{\varrho}_{\text{mass density}} = \sqrt{g}\,N\,T^{00}$, $c\underbrace{j^i}_{\text{mass-current density}} = -\sqrt{g}\,T_i^{\ 0}$, $N^i c = -4A^i$

(T_{ij} not needed, c.f. also ansatz for g_{ij})

$$\varrho_{eff} = \frac{\varrho}{1+\varphi/2c^2} + \frac{2}{c^2}\vec{j}\cdot\vec{A}$$
$$\approx \varrho + \frac{1}{2c^2}(-\varrho\varphi + 4\vec{j}\cdot\vec{A}) + \frac{1}{4c^4}\varrho\varphi^2$$

(potential energy contributes to mass & nonlinearity)
(sign, factor 4 ~ spin 2)

Obviously: $\varphi = O(c^0)$, $\vec{A} = O(c^{-2})$

Comparison with electrodynamics:

gauge condition: $4c^2\,\vec{\nabla}\cdot\vec{A} + 3\dot{\varphi} = 0$

gravitoelectric field: $\vec{E} = -\vec{\nabla}\varphi - \dot{\vec{A}}$

gravitomagnetic field: $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\dot{\vec{B}}$$
$$\vec{\nabla}\cdot\vec{E} = 4\pi G\,\varrho_{eff} + \boxed{\frac{3}{4c^2}\ddot{\varphi}}$$
$$\vec{\nabla}\times\vec{B} = \frac{4\pi G}{c^2}\vec{j} + \frac{1}{c^2}\dot{\vec{E}}$$

ADM Hamiltonian:

- energy of the system, $H_{ADM} = \int d^3x\, \varrho_{eff}\, c^2$
- expressed in terms of canonical variables

(can. var. from source terms)

III. point-masses and dipoles (spin)

point-masses similar to point-charges:

$$\varrho = m_{1\,\mathrm{rel}} \delta_1 + m_{2\,\mathrm{rel}} \delta_2$$

$$= \sqrt{\underset{\uparrow \text{rest-mass}}{m_1^2} + \frac{1}{c^2}\gamma^{ij} p_{1i} p_{1j}} + (1 \leftrightarrow 2)$$

with $\delta_a = \delta(\vec{x} - \vec{q}_a)$

$g_{ij}\gamma^{jk} = \delta_j^k$

$\hookrightarrow \gamma^{ij} = \left(1 + \frac{\varphi}{2c^2}\right)^{-4} \delta_{ij}$

similar: $\vec{j} = \vec{p}_1 \delta_1 + \vec{p}_2 \delta_2$

canonical variables: $\{q_a^i, p_{bj}\} = \delta_{ab}\delta_{ij}$

~~formal~~ expansion of ϱ:

$$\varrho = \varrho_{(0)} c^0 + \varrho_{(2)} c^{-2} + \varrho_{(4)} c^{-4} + \dots$$

$$\varrho_{(0)} = m_1 \delta_1 + m_2 \delta_2$$

$$\varrho_{(2)} = \frac{\vec{p}_1^{\,2}}{2m_1}\delta_1 + \frac{\vec{p}_2^{\,2}}{2m_2}\delta_2$$

$$\varrho_{(4)} = -\frac{(\vec{p}_1^{\,2})^2}{8m_1^3}\delta_1 - \frac{\vec{p}_1^{\,2}}{m_1}\cdot\varphi_{(0)}\delta_1 + (1 \leftrightarrow 2)$$

gives rise to expansion of φ:

$$\varphi = \varphi_{(0)} c^0 + \varphi_{(2)} c^{-2} + \dots, \quad \text{e.g. } \Delta\varphi_{(0)} = -4\pi G \varrho_{(0)}$$

$\hookrightarrow \varphi_{(0)} = \frac{G m_1}{r_1} + \frac{G m_2}{r_2}, \quad r_a = |\vec{x} - \vec{q}_a|$

also: $\vec{j} = \vec{j}_{(0)} c^0, \ \vec{A} = \vec{A}_{(2)} c^{-2} + \dots$

$\hookrightarrow$ Handout

expansion of H_{ADM}:

$$H_{ADM} = H_0 + H_N + H_{1PN} + \dots$$

$$H_0 = \int d^3x\, \varrho_{(0)} c^2 = m_1 c^2 + m_2 c^2$$

$$H_N = \int d^3x \left(\varrho_{(2)} c^{-2} + \frac{1}{2c^2}\varrho_{(0)}\varphi_{(0)}\right)\cdot c^2$$

~~[reminder: $\vec{A} = O(c^{-2})$]~~

$$= \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_1} - \frac{G m_1 m_2}{r_{12}}$$

$r_{12} = |\vec{q}_1 - \vec{q}_2|$

Drop divergent integrals:

$$\int d^3x\, m_1\delta_1 \cdot \frac{G m_1}{r_1} \to 0$$

$$H_{1PN} = c^{-2}\int d^3x \left(\varrho_{(4)} - \frac{1}{2}\varrho_{(2)}\varphi_{(0)} - \frac{1}{2}\varrho_{(0)}\varphi_{(2)} + 2\vec{j}_{(0)}\cdot\vec{A}_{(2)} + \frac{1}{4}\varrho_{(0)}\varphi_{(0)}^2\right)$$

$\hookrightarrow$ Handout

Spin: current dipole, i.e.

$$\vec{j} = \vec{j}_{(0)} = \vec{p}_1 \delta_1 + \frac{1}{2}\vec{S}_1 \times \vec{\nabla}\delta_1 + (1 \leftrightarrow 2)$$

[compare with magnetic dipole]

we also need mass-dipole contributions! (due to can. center, 4dim Th., SSC)

1PN: replace $\vec{p}_a \delta_a$ by $\vec{p}_a \delta_a + \frac{1}{2}\vec{S}_a \times \vec{\nabla}\delta_a$ in ρ

$$\rightsquigarrow \rho^{spin}_{(2)} = \frac{1}{2m_1}\vec{p}_1 \cdot (\vec{S}_1 \times \vec{\nabla}\delta_1) + (1 \leftrightarrow 2)$$

$$\cancel{\rho^{spin}_{(4)} = -\frac{1}{4m_1^3}\vec{p}_1^{\,2}\,\vec{p}_1 \cdot (\vec{S}_1 \times \vec{\nabla}\delta_1)} - \left(\frac{1}{4m_1^3}\vec{p}_1^{\,2} + \frac{1}{m_1}\varphi_{(0)}\right)$$

$$\rho^{spin}_{(4)} = +\frac{1}{m_1}\varphi_{(0)}\,\vec{p}_1 \cdot (\vec{S}_1 \times \vec{\nabla}\delta_1) \underbrace{+ \frac{1}{4m_1^3}\vec{p}_1^{\,2}\,\vec{p}_1 \cdot (\vec{S}_1 \times \vec{\nabla}\delta_1)} + (1 \leftrightarrow 2)$$

$$= \frac{\partial}{\partial x^k}\left(\frac{\vec{p}_1^{\,2}}{4m_1^3}\varepsilon_{ijk}\,p_{1i}\,S_{1j}\,\delta_1\right)$$

$= (td) =$ total divergence (can be dropped)

all variables are still canonical, now also

$$\{S_a^i, S_a^j\} = \varepsilon_{ijk} S_a^k$$

IV gravitational spin-orbit interaction

spin-orbit interaction Hamiltonian H_{SO}^{LO} follows from H_{1PN} (exercise):

$$H_{SO}^{LO} = \frac{G}{c^2 r_{12}^2} (\vec{S}_1 \times \vec{n}_{12}) \cdot \left[\frac{3m_2}{2m_1} \vec{p}_1 - 2\vec{p}_2 \right] + (1 \leftrightarrow 2)$$

Remark: $|\vec{S}_1| \sim \frac{Gm^2}{c} a$

$a = 0 \ldots 1$, dimensionless Kerr-parameter

$\hookrightarrow H_{SO}^{LO}$ is at 1.5PN for $a=1$!

equations of motion:

$$\dot{q}_a^i = \{q_a^i, H\} = +\frac{\partial H}{\partial p_{ai}}$$

$$\dot{p}_{ai} = \{p_{ai}, H\} = -\frac{\partial H}{\partial q_a^i}$$

$$\dot{S}_a^i = \{S_a^i, H\} = \varepsilon_{ijk} \frac{\partial H}{\partial S_a^j} S_a^k = \left(\frac{\partial H}{\partial \vec{S}_a} \times \vec{S} \right)^i$$

$\hookrightarrow$ spin precession equation

center of mass system: $\vec{p}_1 + \vec{p}_2 = 0$

new canonical variables:

$$\vec{p} = \vec{p}_1 = -\vec{p}_2, \quad \vec{q} = \vec{q}_1 - \vec{q}_2$$

$$\{q_i, p^j\} = \delta_{ij}$$

$$H_{SO}^{LO} = \vec{\Omega}_1 \cdot \vec{S}_1 + \vec{\Omega}_2 \cdot \vec{S}_2$$

$$\vec{\Omega}_a = \frac{G}{c^2 q^3} \left(\frac{3m_2}{2m_1} + 2 \right) \vec{L}$$

$$\vec{L} = \vec{q} \times \vec{p}, \quad q = |\vec{q}|$$

$$\dot{\vec{S}}_1 = \vec{\Omega}_1 \times \vec{S}_1$$

Spin is precessing around $\vec{L}$

arXiv:gr-qc/0407116

$$\frac{Dp_\mu}{d\tau} = -\frac{1}{2} R_{\mu\rho\alpha\beta} u^\rho S^{\alpha\beta}$$

$$\frac{DS^{\mu\nu}}{d\tau} = 2p^{[\mu} u^{\nu]} \approx 0$$

$$S^{\mu\nu} p_\nu = 0$$

Dictionary:

$$\varphi = \frac{c^4}{4}\,\phi$$

$$A^i = \frac{c}{2} V^i = -\frac{c}{4} N^i$$

$$\frac{\dot{\Phi}}{c} = -2\,\tilde{\pi}^i{}_{,i} = -\frac{8}{3} V^i{}_{,i} \quad \rightsquigarrow \quad 3\dot{\varphi} + 4c^2\,\vec{\nabla}\cdot\vec{A} = 0$$