Canonical Formulation of Spin

within the ADM Formalism

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Video seminar of the SFB/TR7, May 11th, 2009

Outline

1. Aspects of the ADM Formalism
   - (3+1)-Decomposition
   - ADM Canonical Formalism
2. Hamiltonians from Tulczyjew's Stress-Energy Tensor
   - Stress-Energy Tensor in Canonical Variables
   - Results
3. Higher Orders in Spin and the NLO S_1^2 Hamiltonian
   - Hamiltonians from the Poincaré Algebra
   - The Stress-Energy Tensor with Quadrupole
   - The NLO S_1^2 Hamiltonian
4. Higher Post-Newtonian Orders

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- Decomposition of the field equations:
  - Constraint equations:

$$0 = \mathscr{H} \equiv -\frac{1}{16\pi\sqrt{\gamma}}\left[\gamma \mathrm{R} + \frac{1}{2}\left(\gamma_{ij}\pi^{ij}\right)^2 - \gamma_{ij}\gamma_{kl}\pi^{ik}\pi^{jl}\right] + \mathscr{H}^{\text{matter}}$$

$$0 = \mathscr{H}_i \equiv \frac{1}{8\pi}\gamma_{ij}\pi^{jk}{}_{;k} + \mathscr{H}_i^{\text{matter}}$$

  - Evolution equations:

$$\begin{aligned}
\gamma_{ij,0} &= 2N\gamma^{-1/2}(\pi_{ij} - \tfrac{1}{2}\gamma_{ij}\gamma_{kl}\pi^{kl}) + N_{i;j} + N_{j;i} \\
\pi^{ij}{}_{,0} &= -N\sqrt{\gamma}(\mathrm{R}^{ij} - \tfrac{1}{2}\gamma^{ij}\mathrm{R}) + \tfrac{1}{2}N\gamma^{-1/2}\gamma^{ij}(\pi^{mn}\pi_{mn} - \tfrac{1}{2}(\gamma_{mn}\pi^{mn})^2) \\
&\quad - 2N\gamma^{-1/2}(\gamma_{mn}\pi^{im}\pi^{nj} - \tfrac{1}{2}\gamma_{mn}\pi^{mn}\pi^{ij}) + \sqrt{\gamma}(N^{;ij} - \gamma^{ij}N^{;m}{}_{;m}) \\
&\quad + (\pi^{ij}N^m)_{;m} - N^i{}_{;m}\pi^{mj} - N^j{}_{;m}\pi^{mi} + \tfrac{1}{2}N\gamma^{im}\gamma^{nj}\sqrt{\gamma}T_{mn}
\end{aligned}$$

- Source terms are related to the stress-energy tensor $T^{\mu\nu}$ by:

$$\mathscr{H}^{\text{matter}} \equiv \sqrt{\gamma}T_{\mu\nu}n^\mu n^\nu$$

$$\mathscr{H}_i^{\text{matter}} \equiv -\sqrt{\gamma}T_{i\nu}n^\nu$$

ADM Canonical Formalism

- Gauge independent Hamiltonian:

$$H[x_a^i, p_{ai}, \gamma_{ij}, \pi^{ij}] = \int d^3\mathbf{x}(N\mathscr{H} - N^i\mathscr{H}_i) + E[\gamma_{ij}]$$

$$E[\gamma_{ij}] = \frac{1}{16\pi}\oint d^2 s_i(\gamma_{ij,j} - \gamma_{jj,i})$$

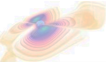
- Hamiltonian in ADMTT gauge (ADM Hamiltonian)
  $\hat{=}$ ADM energy depending on canonical variables:

$$H_{\mathrm{ADM}} = E[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}, \pi_{\mathrm{TT}}^{ij}] = -\frac{1}{16\pi}\int \mathrm{d}^3\mathbf{x}\,\Delta\phi$$

$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$

- Matter only Hamiltonian: Elimination of h_{ij}^{TT} and π_{TT}^{ij}.

Outline



Spin in GR

- Stress-energy tensor density in covariant SSC, $S^{\mu\nu} u_\nu = 0$:

$$\sqrt{-g}\; T^{\mu\nu} = \int d\tau \left[m u^\mu u^\nu \delta_{(4)} - (S^{\alpha(\mu} u^{\nu)} \delta_{(4)})_{||\alpha} \right]$$

$$\delta_{(4)} \equiv \delta(x - q(\tau))$$

- EOM follow from $T^{\mu\nu}{}_{||\nu} = 0$:

$$\frac{DS^{\mu\nu}}{d\tau} = 0\,, \quad m\frac{Du_\mu}{d\tau} = \frac{1}{2} S^{\lambda\nu} u^\gamma R^{(4)}_{\mu\gamma\nu\lambda}$$

Identification of Canonical Variables

- Calculate $\mathcal{H}_i^{\text{matter}}$:

$$\mathcal{H}_i^{\text{matter}} = -\sqrt{\gamma} T_{i\nu} n^{\nu}$$

- Define canonical momentum p_i as:

$$p_i = \int d^3\mathbf{x} \, \mathcal{H}_i^{\text{matter}}$$

- Define spin $\hat{S}_{ij} = e_{i(k)} e_{j(l)} \varepsilon_{klm} S_{(m)}$ such that $\mathbf{S}^2 = \text{const.}$ and

$$J_{ij} = z^i p_j - z^j p_i + \varepsilon_{ijm} S_{(m)} = \int d^3\mathbf{x} \, (x^i \mathcal{H}_j^{\text{matter}} - x^j \mathcal{H}_i^{\text{matter}})$$

- Go over to canonical position variable $\mathbf{z}$ by a Lie shift (such that one has the Newton-Wigner SSC in flat space).

NLO Spin-Orbit Hamiltonian

First derived: Damour, Jaranowski, and Schäfer (2008)

$$
\begin{aligned}
H_{SO}^{NLO} = &-\frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^2}\left[\frac{5 m_2 \mathbf{p}_1^2}{8 m_1^3} + \frac{3(\mathbf{p}_1\cdot\mathbf{p}_2)}{4 m_1^2} - \frac{3\mathbf{p}_2^2}{4 m_1 m_2} \right. \\
&\left. \qquad + \frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{4 m_1^2} + \frac{3(\mathbf{p}_2\cdot\mathbf{n}_{12})^2}{2 m_1 m_2}\right] \\
&+ \frac{((\mathbf{p}_2 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^2}\left[\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1 m_2}\right] \\
&+ \frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot \mathbf{p}_2)}{r_{12}^2}\left[\frac{2(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1 m_2} - \frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})}{4 m_1^2}\right] \\
&- \frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^3}\left[\frac{11 m_2}{2} + \frac{5 m_2^2}{m_1}\right] \\
&+ \frac{((\mathbf{p}_2 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^3}\left[6 m_1 + \frac{15 m_2}{2}\right] + (1 \leftrightarrow 2)
\end{aligned}
$$

NLO Spin$_1$-Spin$_2$ Hamiltonian

$$
\begin{aligned}
H^{\text{NLO}}_{S_1S_2} = \frac{1}{2m_1 m_2 r_{12}^3}\Big[& \tfrac{3}{2}((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_2 \times \mathbf{S}_2)\cdot\mathbf{n}_{12}) + \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{p}_2) \\
& + 6((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1 \times \mathbf{S}_2)\cdot\mathbf{n}_{12}) - \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_1) \\
& - 15(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + (\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{p}_2) \\
& - 3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{p}_2) + 3(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) - 3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})\Big] \\
+ \frac{3}{2m_1^2 r_{12}^3}\Big[& -((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1 \times \mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
& + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{n}_{12})\Big] \\
+ \frac{3}{2m_2^2 r_{12}^3}\Big[& -((\mathbf{p}_2 \times \mathbf{S}_2)\cdot\mathbf{n}_{12})((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12}) \\
& + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})\Big] \\
+ \frac{6(m_1+m_2)}{r_{12}^4}\Big[& (\mathbf{S}_1\cdot\mathbf{S}_2) - 2(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})\Big]
\end{aligned}
$$

NLO Center of Mass

$$
\begin{aligned}
\mathbf{G}_{\mathrm{SO}}^{\mathrm{NLO}} = & -\sum_a \frac{\mathbf{p}_a^2}{8m_a^3}(\mathbf{p}_a \times \mathbf{S}_a) \\
& + \sum_a \sum_{b \neq a} \frac{m_b}{4m_a r_{ab}} \left[((\mathbf{p}_a \times \mathbf{S}_a) \cdot \mathbf{n}_{ab}) \frac{5\mathbf{z}_a + \mathbf{z}_b}{r_{ab}} - 5(\mathbf{p}_a \times \mathbf{S}_a) \right] \\
& + \sum_a \sum_{b \neq a} \frac{1}{r_{ab}} \left[\frac{3}{2}(\mathbf{p}_b \times \mathbf{S}_a) - \frac{1}{2}(\mathbf{n}_{ab} \times \mathbf{S}_a)(\mathbf{p}_b \cdot \mathbf{n}_{ab}) \right. \\
& \qquad \left. - ((\mathbf{p}_a \times \mathbf{S}_a) \cdot \mathbf{n}_{ab}) \frac{\mathbf{z}_a + \mathbf{z}_b}{r_{ab}} \right]
\end{aligned}
$$

$$
\mathbf{G}_{\mathrm{SS}}^{\mathrm{NLO}} = \frac{1}{2} \sum_a \sum_{b \neq a} \left\{ [3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b)] \frac{\mathbf{z}_a}{r_{ab}^3} + (\mathbf{S}_b \cdot \mathbf{n}_{ab}) \frac{\mathbf{S}_a}{r_{ab}^2} \right\}
$$

$\Rightarrow$ Poincaré algebra is fulfilled.

Outline

Global Poincaré Invariance

The algebra of global Poincaré invariance reads

$$\{P_i, P_j\} = 0\,, \qquad \{P_i, H\} = 0\,, \qquad \{J_i, H\} = 0\,,$$
$$\{J_i, P_j\} = \varepsilon_{ijk} P_k\,, \qquad \{J_i, J_j\} = \varepsilon_{ijk} J_k\,, \qquad \{J_i, G_j\} = \varepsilon_{ijk} G_k\,,$$
$$\{G_i, P_j\} = H\delta_{ij}\,, \qquad {\color{red}\{G_i, H\} = P_i}\,, \qquad \{G_i, G_j\} = -\varepsilon_{ijk} J_k\,,$$

with

$$P_i = \sum_a p_{ai}\,, \quad J_i = \sum_a \left[\varepsilon_{ijk} z_a^j p_{ak} + S_{a(i)}\right]\,.$$

Hamiltonians from the Poincaré Algebra

Hergt and Schäfer (2008)

The full Hamiltonian up to 2PN enters the Poincaré algebra:

$$H = H_N + H_{1PN} + H_{2PN} + H_{SO}^{1PN} + H_{SO}^{2PN} + H_{S^2} + H_{S^3 p} + H_{S^2 p^2} + H_{S^4}$$

- Source terms in canonical variables sufficient for $H_{S_2^2 S_1 p_1}$, $H_{S_2^3 p_1}$, $H_{S_1^3 p_2}$, $H_{S_1^2 S_2 p_2}$, $H_{S_1^2 S_2^2}$, $H_{S_1 S_2^3}$, and $H_{S_2 S_1^3}$ were obtained from the Kerr-metric in ADM coordinates (HS 2007).
- Ansatzes for $H_{S_1^2 p^2}$, $H_{S_2^2 p^2}$, $H_{S_1^3 p_1}$, $H_{S_2^3 p_2}$, $H_{S_1^2 S_2 p_1}$, $H_{S_2^2 S_1 p_2}$, $H_{S_1^4}$, and $H_{S_2^4}$ are fixed up to canonical transformation by $\{G_i, H\} = P_i$.
- The static (linear momentum independent) part of the Hamilton constraint is needed to fix these remaining degrees of freedom.

The Stress-Energy Tensor with Quadrupole

- Stress-energy tensor density with quadrupole has the structure:

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \left[t^{\mu\nu}\delta_{(4)} + (t^{\mu\nu\alpha}\delta_{(4)})_{||\alpha} + (t^{\mu\nu\alpha\beta}\delta_{(4)})_{||\alpha\beta} \right]$$

- Getting expressions for the $t^{\mu\nu\cdots}$ from $T^{\mu\nu}_{\ \ ||\nu} = 0$:
  - Dixon's work: Complicated definitions.
  - Tulczyjew's theorems: Complicated calculation.

Ansatz for the Static Source Terms

$$\mathscr{H}^{\text{matter}}_{S_1^2,\,\text{static}} = \frac{c_1}{m_1}\left(I_1^{ij}\delta_1\right)_{;ij} + \frac{c_2}{m_1}\mathrm{R}_{ij}I_1^{ij}\delta_1 + \frac{c_3}{m_1}\mathbf{S}_1^2\left(\gamma^{ij}\delta_1\right)_{;ij} + \frac{c_4}{m_1}\mathrm{R}\mathbf{S}_1^2\delta_1$$
$$+ \frac{1}{8m_1} g_{mn}\gamma^{pj}\gamma^{ql}\gamma^{mi}{}_{,p}\gamma^{nk}{}_{,q}\hat{S}_{1\,ij}\hat{S}_{1\,kl}\delta_1$$
$$+ \frac{1}{4m_1}\left(\gamma^{ij}\gamma^{mn}\gamma^{kl}{}_{,m}\hat{S}_{1\,ln}\hat{S}_{1\,jk}\delta_1\right)_{,i}$$

- This ansatz is 3-dim. covariant, as p_i is not:

$$p_i = \int d^3\mathbf{x}\,\mathscr{H}^{\text{matter}}_i = mv_i - \frac{1}{2}g_{ij}\gamma^{lm}\gamma^{kj}{}_{,m}\hat{S}_{kl} + \mathscr{O}(p^2) + \mathscr{O}(\hat{S}^2)$$

- Terms like $I_1^{ij}{}_{;k}\delta_1$ or $I_1^{ij}\delta_{1;k}$ can not appear.
- γ_{ij} for Kerr $\Rightarrow c_1 = -\frac{1}{2}$.
- N for Kerr $\Rightarrow c_2 = 0$.
- c_3 and c_4 do not contribute to the Hamiltonian.

NLO Spin_1-Spin_1 Hamiltonian

$$
\begin{aligned}
H^{\text{NLO}}_{S_1^2} = \frac{1}{r_{12}^3}\Bigg[& \frac{m_2}{4m_1^3}(\mathbf{p}_1\cdot\mathbf{S}_1)^2 + \frac{3m_2}{8m_1^3}(\mathbf{p}_1\cdot\mathbf{n}_{12})^2\,\mathbf{S}_1^2 - \frac{3m_2}{8m_1^3}\mathbf{p}_1^2(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 \\
& - \frac{3m_2}{4m_1^3}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{S}_1) - \frac{3}{4m_1 m_2}\mathbf{p}_2^2\mathbf{S}_1^2 \\
& + \frac{9}{4m_1 m_2}\mathbf{p}_2^2(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 + \frac{3}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{p}_2)\,\mathbf{S}_1^2 \\
& - \frac{9}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 + \frac{3}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})\,\mathbf{S}_1^2 \\
& - \frac{3}{2m_1^2}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{S}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12}) \\
& + \frac{3}{m_1^2}(\mathbf{p}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{S}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12}) \\
& - \frac{15}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 \Bigg] \\
- \frac{m_2}{r_{12}^4}\Bigg[& \frac{9}{2}(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 - \frac{5}{2}\mathbf{S}_1^2 + \frac{7m_2}{m_1}(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 - \frac{3m_2}{m_1}\mathbf{S}_1^2 \Bigg]
\end{aligned}
$$

Outline

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Higher Post-Newtonian Orders

with Han Wang, unpublished

- Need spin corrections to canonical field momentum:

$$\pi^{ij}_{\text{can}} = \pi^{ij}_{\text{field}} + \pi^{ij}_{\text{spin}} \,,$$

$$\pi^{ij}_{\text{field}} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{kl} \,.$$

- Choose π^{ij}_{spin} such that:

$$P_i = \sum_a p_{ai} - \frac{1}{16\pi}\int d^3x \, \pi^{kl\text{TT}}_{\text{can}} h^{\text{TT}}_{kl,i}$$

$$J_{ij} = \sum_a (z^i_a p_{aj} - z^j_a p_{ai}) + \sum_a S_{a(i)(j)} - \frac{1}{16\pi}\int d^3x \, (x^i \pi^{kl\text{TT}}_{\text{can}} h^{\text{TT}}_{kl,j} - x^j \pi^{kl\text{TT}}_{\text{can}} h^{\text{TT}}_{kl,i}) + 2\frac{1}{16\pi}\int d^3x \, (\pi^{ik\text{TT}}_{\text{can}} h^{\text{TT}}_{kj} - \pi^{jk\text{TT}}_{\text{can}} h^{\text{TT}}_{ki})$$

- Got Hamiltonian for field evolution at formal 3.5PN.
- Checked 1PN energy flux (Kidder 1995).
- Formal 3PN order? Extension to S_1^2?

Thank you for your attention!