Spin effects in the post-Newtonian approximation:

Quadrupole deformation of Neutron stars and three-body interactions

J. Hartung S. Hergt G. Schäfer J. Steinhoff

since 1558

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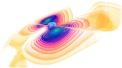
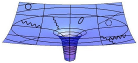
Video seminar of the SFB/TR7, November 22nd, 2010





DFG : SFB/TR7 "Gravitational Wave Astronomy" and GRK 1523

Outline

1. Three-body interactions with spin:
   NLO Spin-Orbit and NLO Spin(a)-Spin(b) Hamiltonians

   J. Hartung and J. Steinhoff
   Phys. Rev. D, submitted, arXiv:1011.1179

2. Action approach to canonical formulation of spin in GR

   J. Steinhoff and G. Schäfer
   Europhys. Lett. **87**, 50004 (2009)

3. Quadrupole deformation of Neutron stars due to Spin:
   NLO Spin(1)-Spin(1) Hamiltonian

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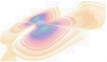
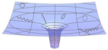




Three-body interactions

- $H^{\mathrm{ADM}} \,\hat{=}\,$ ADM energy expressed in terms of canonical variables after field constraints are solved in the ADMTT gauge.
- Canonical variables are denoted by a hat $\hat{}$.
- Only pairwise two-body interactions in the Newtonian case:

$$H^{\mathrm{N}} = \sum_a \frac{\hat{\mathbf{p}}_a^2}{2m_a} - \frac{1}{2}\sum_a \sum_{b\neq a} \frac{G m_a m_b}{\hat{r}_{ab}} \tag{1}$$

- At 1PN three-body interactions appear:

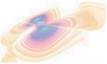
$$H^{\mathrm{1PN}} = -\sum_a \frac{(\hat{\mathbf{p}}_a^2)^2}{8m_a^3} + \sum_a \sum_{b\neq a} \frac{G}{\hat{r}_{ab}} \left[-\frac{3m_b}{2m_a}\hat{\mathbf{p}}_a^2 + \frac{1}{4}\left(7(\hat{\mathbf{p}}_a\hat{\mathbf{p}}_b) + (\hat{\mathbf{n}}_{ab}\hat{\mathbf{p}}_a)(\hat{\mathbf{n}}_{ab}\hat{\mathbf{p}}_b)\right)\right] + \sum_a \sum_{b\neq a} \frac{G^2 m_a^2 m_b}{2\hat{r}_{ab}^2} + \sum_a \sum_{b\neq a} \sum_{c\neq a,b} \frac{G^2 m_a m_b m_c}{2\hat{r}_{ab}\hat{r}_{ac}} \tag{2}$$

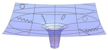
- The leading order in spin is only a sum of two-body interactions.
- Three-body interactions with spin appear at next-to-leading order (NLO).

NLO Spin-Orbit, two-body interaction part

Hamiltonian first derived: Damour, Jaranowski, Schäfer (2008)
See also: Tagoshi, Ohashi, Owen (2001); Faye, Blanchet, Buonanno (2006)

$$
\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{SO},[2]} = \sum_a \sum_{b \neq a} \Bigg(& -G \frac{((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab})}{\hat{r}_{ab}^2} \left[\frac{5 m_b \hat{\mathbf{p}}_a^2}{8 m_a^3} + \frac{3 (\hat{\mathbf{p}}_a \cdot \hat{\mathbf{p}}_b)}{4 m_a^2} - \frac{3 \hat{\mathbf{p}}_b^2}{4 m_a m_b} \right. \\
& \left. + \frac{3 (\hat{\mathbf{p}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab})}{4 m_a^2} + \frac{3 (\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab})^2}{2 m_a m_b} \right] \\
& + G \frac{((\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab})}{\hat{r}_{ab}^2} \left[\frac{(\hat{\mathbf{p}}_a \cdot \hat{\mathbf{p}}_b)}{m_a m_b} + \frac{3 (\hat{\mathbf{p}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab})}{m_a m_b} \right] \\
& + G \frac{((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{p}}_b)}{\hat{r}_{ab}^2} \left[\frac{2 (\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab})}{m_a m_b} - \frac{3 (\hat{\mathbf{p}}_a \cdot \hat{\mathbf{n}}_{ab})}{4 m_a^2} \right] \\
& - G^2 \frac{((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab})}{\hat{r}_{ab}^3} \left[\frac{11 m_b}{2} + \frac{5 m_b^2}{m_a} \right] \\
& + G^2 \frac{((\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab})}{\hat{r}_{ab}^3} \left[6 m_a + \frac{15 m_b}{2} \right] \Bigg)
\end{aligned}
$$



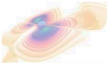


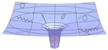
NLO Spin-Orbit, three-body interaction part

Hartung, Steinhoff, arXiv:1011.1179

$$
\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{SO},[3]} = \sum_a \sum_{b\neq a} \sum_{c\neq a,b} \Bigg(& G^2 m_a \Bigg[\frac{1}{\hat{s}_{abc}} \Bigg\{ \Bigg(\frac{3}{\hat{r}_{ab}\hat{r}_{ac}} - \frac{6}{\hat{r}_{ab}\hat{r}_{bc}} - \frac{3}{\hat{r}_{ac}\hat{r}_{bc}} - \frac{3\hat{r}_{ab}}{\hat{r}_{ac}^2\hat{r}_{bc}} + \frac{3\hat{r}_{bc}}{\hat{r}_{ab}\hat{r}_{ac}^2} \Bigg) ((\hat{\mathbf{n}}_{ac}\times\hat{\mathbf{p}}_b)\,\hat{\mathbf{S}}_c) \\
& + \Bigg(-\frac{8}{\hat{r}_{bc}^2} + \frac{8}{\hat{r}_{ab}\hat{r}_{ac}} - \frac{4}{\hat{r}_{ab}\hat{r}_{bc}} - \frac{4}{\hat{r}_{ac}\hat{r}_{bc}} - \frac{4\hat{r}_{ab}}{\hat{r}_{ac}\hat{r}_{bc}^2} - \frac{4\hat{r}_{ac}}{\hat{r}_{ab}\hat{r}_{bc}^2} \Bigg) ((\hat{\mathbf{n}}_{bc}\times\hat{\mathbf{p}}_b)\,\hat{\mathbf{S}}_c) \Bigg\} \\
& + \frac{((\hat{\mathbf{n}}_{ac}\times\hat{\mathbf{n}}_{bc})\,\hat{\mathbf{S}}_c)}{\hat{s}_{abc}^2} \Bigg\{ \Bigg(\frac{4}{\hat{r}_{ab}} - \frac{2\hat{r}_{bc}}{\hat{r}_{ab}\hat{r}_{ac}} + \frac{2\hat{r}_{ac}}{\hat{r}_{ab}^2} - \frac{2\hat{r}_{bc}^2}{\hat{r}_{ab}^2\hat{r}_{ac}} + \frac{8\hat{r}_{ab}}{\hat{r}_{ac}^2} + \frac{7\hat{r}_{bc}}{\hat{r}_{ac}^2} - \frac{2\hat{r}_{bc}^2}{\hat{r}_{ab}\hat{r}_{ac}^2} + \frac{\hat{r}_{bc}}{\hat{r}_{ab}^2} - \frac{\hat{r}_{bc}^3}{\hat{r}_{ab}^2\hat{r}_{ac}^2} \Bigg) (\hat{\mathbf{n}}_{ab}\hat{\mathbf{p}}_b) \\
& + \Bigg(\frac{12}{\hat{r}_{ab}} - \frac{2}{\hat{r}_{ac}} + \frac{5}{\hat{r}_{bc}} - \frac{2\hat{r}_{ab}}{\hat{r}_{ac}^2} + \frac{7\hat{r}_{bc}}{\hat{r}_{ac}^2} - \frac{2\hat{r}_{ab}}{\hat{r}_{ac}\hat{r}_{bc}} + \frac{6\hat{r}_{ac}}{\hat{r}_{ab}\hat{r}_{bc}} - \frac{\hat{r}_{ab}^2}{\hat{r}_{ac}^2\hat{r}_{bc}} + \frac{8\hat{r}_{bc}^2}{\hat{r}_{ab}\hat{r}_{ac}^2} \Bigg) (\hat{\mathbf{n}}_{bc}\hat{\mathbf{p}}_b) + \frac{16(\hat{\mathbf{n}}_{ac}\hat{\mathbf{p}}_b)}{\hat{r}_{ab}} \Bigg\} \Bigg] \\
& + G^2 \frac{m_a m_b}{m_c} \Bigg[\frac{1}{\hat{s}_{abc}^2} \Bigg\{ \frac{1}{\hat{r}_{ab}^3} \Bigg(-2\hat{r}_{ac}^2 + \hat{r}_{bc}^2 - \frac{3}{2}\hat{r}_{ac}\hat{r}_{bc} + \frac{1}{2}\frac{\hat{r}_{bc}^3}{\hat{r}_{ac}} \Bigg) + \frac{1}{\hat{r}_{ab}^2} \Bigg(-4\hat{r}_{ac} + \hat{r}_{bc} + \frac{\hat{r}_{bc}^2}{\hat{r}_{ac}} \Bigg) - \frac{1}{\hat{r}_{ac}} \\
& - \frac{1}{\hat{r}_{ab}} \Bigg(2 + \frac{1}{2}\frac{\hat{r}_{bc}}{\hat{r}_{ac}} \Bigg) \Bigg\} (\hat{\mathbf{n}}_{ac}\hat{\mathbf{p}}_c)((\hat{\mathbf{n}}_{ac}\times\hat{\mathbf{n}}_{bc})\,\hat{\mathbf{S}}_c) + \frac{1}{\hat{s}_{abc}} \Bigg\{ \frac{1}{\hat{r}_{ab}^3} \Bigg(\frac{1}{8}\hat{r}_{ac} - \frac{5}{8}\hat{r}_{bc} + \frac{3}{4}\frac{\hat{r}_{ac}^2}{\hat{r}_{bc}} - \frac{1}{8}\frac{\hat{r}_{bc}^2}{\hat{r}_{ac}} - \frac{1}{8}\frac{\hat{r}_{bc}^3}{\hat{r}_{ac}^2} \Bigg) \\
& + \frac{1}{\hat{r}_{ab}^2} \Bigg(-\frac{5}{8} + \frac{3}{4}\frac{\hat{r}_{ac}}{\hat{r}_{bc}} - \frac{1}{8}\frac{\hat{r}_{bc}^2}{\hat{r}_{ac}^2} \Bigg) + \frac{1}{\hat{r}_{ab}} \Bigg(\frac{3}{8\hat{r}_{ac}} - \frac{1}{\hat{r}_{bc}} + \frac{3}{8}\frac{\hat{r}_{bc}}{\hat{r}_{ac}^2} \Bigg) - \frac{\hat{r}_{ab}}{4\hat{r}_{ac}^2\hat{r}_{bc}} - \frac{1}{4\hat{r}_{ac}\hat{r}_{bc}} \\
& + \frac{1}{8\hat{r}_{ac}^2} \Bigg\} ((\hat{\mathbf{n}}_{ac}\times\hat{\mathbf{p}}_c)\,\hat{\mathbf{S}}_c) + (a\leftrightarrow b) \Bigg] - \frac{G^2}{\hat{r}_{ab}^2} \Bigg(\frac{5}{\hat{r}_{ac}} + \frac{1}{\hat{r}_{bc}} \Bigg) \frac{m_b m_c}{m_a} ((\hat{\mathbf{n}}_{ab}\times\hat{\mathbf{p}}_a)\,\hat{\mathbf{S}}_a) \Bigg)
\end{aligned}
$$

$$
\hat{s}_{abc} = \hat{r}_{ab} + \hat{r}_{ac} + \hat{r}_{bc}
$$



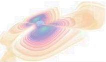


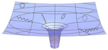
NLO Spin(a)-Spin(b), two-body interaction part

Partial result: Porto, Rothstein (2006). Full result: Steinhoff, Hergt, Schäfer (2008).

$$
\begin{aligned}
H^{\rm NLO}_{S_a S_b,[2]} = \sum_a \sum_{b \neq a} \Bigg(& \frac{G}{4 m_a m_b \hat{r}_{ab}^3} [\tfrac{3}{2} ((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab})((\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_b) \cdot \hat{\mathbf{n}}_{ab}) \\
& + 6((\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab})((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_b) \cdot \hat{\mathbf{n}}_{ab}) - \tfrac{1}{2} (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{p}}_b)(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{p}}_a) \\
& - 15 (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{p}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab}) + (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{p}}_a)(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{p}}_b) \\
& - 3 (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{p}}_a \cdot \hat{\mathbf{p}}_b) + 3 (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{p}}_b)(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{p}}_a \cdot \hat{\mathbf{n}}_{ab}) \\
& + 3 (\hat{\mathbf{S}}_b \cdot \hat{\mathbf{p}}_a)(\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab}) + 3 (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{p}}_a)(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab}) \\
& + 3 (\hat{\mathbf{S}}_b \cdot \hat{\mathbf{p}}_b)(\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{p}}_a \cdot \hat{\mathbf{n}}_{ab}) - 3 (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{S}}_b)(\hat{\mathbf{p}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab}) \\
& + \tfrac{1}{2} (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{S}}_b)(\hat{\mathbf{p}}_a \cdot \hat{\mathbf{p}}_b)] \\
& + \frac{3G}{2 m_a^2 \hat{r}_{ab}^3} [-((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab})((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_b) \cdot \hat{\mathbf{n}}_{ab}) \\
& + (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{S}}_b)(\hat{\mathbf{p}}_a \cdot \hat{\mathbf{n}}_{ab})^2 - (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{p}}_a)(\hat{\mathbf{p}}_a \cdot \hat{\mathbf{n}}_{ab})] \\
& + \frac{6 G^2 m_a}{\hat{r}_{ab}^4} [(\hat{\mathbf{S}}_a \cdot \hat{\mathbf{S}}_b) - 2 (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab})] \Bigg)
\end{aligned}
$$

since 1558





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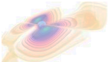
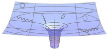
Hartung, Steinhoff, arXiv:1011.1179

$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_aS_b,[3]} = \sum_{a}\sum_{b\neq a}\sum_{c\neq a,b} \frac{G^2 m_c}{\hat{s}_{abc}^2}\Bigg[& ((\hat{\mathbf{n}}_{ac}\times\hat{\mathbf{n}}_{bc})\,\hat{\mathbf{S}}_a)((\hat{\mathbf{n}}_{ac}\times\hat{\mathbf{n}}_{bc})\,\hat{\mathbf{S}}_b)\Bigg\{\frac{1}{\hat{r}_{ac}\hat{r}_{bc}} + \frac{4}{\hat{r}_{ab}\hat{r}_{ac}} + \frac{4}{\hat{r}_{ab}^2}\left(\frac{1}{2}+\frac{\hat{r}_{ac}}{\hat{r}_{bc}}\right) \\
& + \frac{2}{\hat{r}_{ab}^3}\left(2\hat{r}_{ac}+\frac{\hat{r}_{ac}^2}{\hat{r}_{bc}}\right)\Bigg\} + (\hat{\mathbf{n}}_{bc}\hat{\mathbf{S}}_a)(\hat{\mathbf{n}}_{ac}\hat{\mathbf{S}}_b)\Bigg\{-\frac{1}{\hat{r}_{ac}^2} - \frac{1}{\hat{r}_{ac}\hat{r}_{bc}} - \frac{2\hat{r}_{ab}}{\hat{r}_{ac}^2\hat{r}_{bc}} - \frac{\hat{r}_{ab}^2}{2\hat{r}_{ac}^2\hat{r}_{bc}^2} + \frac{\hat{r}_{ac}}{\hat{r}_{ab}\hat{r}_{bc}^2} \\
& + \frac{2}{\hat{r}_{ab}^2}\left(-1+\frac{\hat{r}_{ac}}{\hat{r}_{bc}}+\frac{\hat{r}_{ac}^2}{\hat{r}_{bc}^2}\right) + \frac{1}{\hat{r}_{ab}^3}\left(2\hat{r}_{ac}+\frac{2\hat{r}_{ac}^2}{\hat{r}_{bc}}+\frac{\hat{r}_{ac}^3}{\hat{r}_{bc}^2}\right) + \frac{3(\hat{r}_{ac}+\hat{r}_{bc})^2}{\hat{r}_{ab}^4} + \frac{3(\hat{r}_{ac}+\hat{r}_{bc})^3}{2\hat{r}_{ab}^5}\Bigg\} \\
& + (\hat{\mathbf{n}}_{ac}\hat{\mathbf{S}}_a)(\hat{\mathbf{n}}_{ac}\hat{\mathbf{S}}_b)\Bigg\{-\frac{3}{\hat{r}_{ab}^5}\left(3\hat{r}_{ac}^3+3\hat{r}_{ac}^2\hat{r}_{bc}+\hat{r}_{ac}\hat{r}_{bc}^2+\frac{\hat{r}_{ac}^4}{\hat{r}_{bc}}\right) - \frac{6}{\hat{r}_{ab}^4}\left(2\hat{r}_{ac}^2+\hat{r}_{ac}\hat{r}_{bc}+\frac{\hat{r}_{ac}^3}{\hat{r}_{bc}}\right) \\
& - \frac{1}{\hat{r}_{ab}^3}\left(2\hat{r}_{ac}+2\hat{r}_{bc}+\frac{\hat{r}_{ac}^2}{\hat{r}_{bc}}+\frac{\hat{r}_{bc}^2}{\hat{r}_{ac}}\right) - \frac{2}{\hat{r}_{ab}^2}\left(1-\frac{2\hat{r}_{ac}}{\hat{r}_{bc}}+\frac{\hat{r}_{bc}}{\hat{r}_{ac}}\right) + \frac{1}{\hat{r}_{ab}}\left(\frac{1}{\hat{r}_{ac}}+\frac{4}{\hat{r}_{bc}}\right) + \frac{2}{\hat{r}_{ac}\hat{r}_{bc}}\Bigg\} \\
& + (\hat{\mathbf{n}}_{ac}\hat{\mathbf{S}}_a)(\hat{\mathbf{n}}_{bc}\hat{\mathbf{S}}_b)\Bigg\{-\frac{2}{\hat{r}_{ab}^2} + \frac{\hat{r}_{ac}}{\hat{r}_{ab}^3} + \frac{3(\hat{r}_{ac}+\hat{r}_{bc})^2}{\hat{r}_{ab}^4} + \frac{3(\hat{r}_{ac}+\hat{r}_{bc})^3}{2\hat{r}_{ab}^5}\Bigg\} + (\hat{\mathbf{S}}_a\hat{\mathbf{S}}_b)\Bigg\{\frac{2}{\hat{r}_{ac}^2} - \frac{3}{2\hat{r}_{ac}\hat{r}_{bc}} \\
& + \frac{3}{2}\frac{\hat{r}_{ac}}{\hat{r}_{bc}^3} + \hat{r}_{ab}\left(\frac{3}{2\hat{r}_{ac}^3}+\frac{1}{\hat{r}_{ac}^2\hat{r}_{bc}}\right) - \frac{\hat{r}_{ab}^2}{2\hat{r}_{ac}^2\hat{r}_{bc}^2} - \frac{\hat{r}_{ab}^3}{\hat{r}_{ac}^3\hat{r}_{bc}^2} - \frac{\hat{r}_{ab}^4}{4\hat{r}_{ac}^3\hat{r}_{bc}^3} + \frac{1}{\hat{r}_{ab}}\left(-\frac{2}{\hat{r}_{ac}}+\frac{\hat{r}_{ac}}{\hat{r}_{bc}^2}\right) \\
& + \frac{1}{\hat{r}_{ab}^2}\left(3+\frac{3\hat{r}_{ac}}{\hat{r}_{bc}}-\frac{\hat{r}_{ac}^2}{\hat{r}_{bc}^2}-\frac{\hat{r}_{ac}^3}{\hat{r}_{bc}^3}\right) + \frac{1}{\hat{r}_{ab}^3}\left(\frac{9}{2}\hat{r}_{ac}+\frac{\hat{r}_{ac}^2}{\hat{r}_{bc}}-\frac{\hat{r}_{ac}^3}{\hat{r}_{bc}^2}-\frac{\hat{r}_{ac}^4}{2\hat{r}_{bc}^3}\right)\Bigg\} + (a\leftrightarrow b)\Bigg]
\end{aligned}
$$





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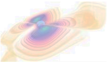
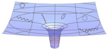




Angular Velocity and Spin

in Newtonian mechanics and special relativity

	Newton	special relativity
body-fixed frame	$x^{[i]} = \Lambda_{[i]j} x^j$	
rotational degrees of freedom	$\Lambda_{[k]i}\Lambda_{[k]j} = \delta_{ij}$	$\eta^{AB}\Lambda_{A\mu}\Lambda_{B\nu} = \eta_{\mu\nu}$
$\hookrightarrow$ supplementary condition		$\Lambda^{[i]\mu} p_\mu = 0$
Angular Velocity	$\Omega^{ij} = \Lambda_{[k]i} \frac{\mathrm{d}\Lambda_{[k]j}}{\mathrm{d}t}$	$\Omega^{\mu\nu} = \Lambda_A{}^\mu \frac{\mathrm{d}\Lambda^{A\nu}}{\mathrm{d}\tau}$
Spin (L: Lagrangian)	$S_{ij} = 2\frac{\partial L}{\partial \Omega^{ij}}$	$S_{\mu\nu} = 2\frac{\partial L}{\partial \Omega^{\mu\nu}}$
$\hookrightarrow$ supplementary condition		$S_{\mu\nu} p^\nu = 0$

since 1558

Remark:

- Angular velocity vector is $\Omega^i = \frac{1}{2}\epsilon_{ijk}\Omega^{jk}$. Analogous for spin.

Action approach with minimal coupling

- Gravitational field is given by a tetrad $e_{a\mu}$:

$$\Lambda_{A\mu}\Lambda^{A}{}_{\nu} = g_{\mu\nu} \quad \rightarrow \quad \Lambda_{Aa}\Lambda^{A}{}_{b} = \eta_{ab} \tag{3}$$

- Minimal coupling:

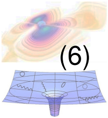
$$e^{a}{}_{\mu} e^{b}{}_{\nu} \Omega^{\mu\nu} = \Omega^{ab} = \Lambda_{A}{}^{a} \frac{D\Lambda^{Ab}}{d\tau} = \Lambda_{A}{}^{a} \left[\frac{d\Lambda^{Ab}}{d\tau} - \Lambda^{A}{}_{c} \omega_{\mu}{}^{cb} u^{\mu} \right] \tag{4}$$

- Supplementary conditions and mass-shell constraint:

$$S_{\mu\nu} p^{\nu} = 0 \,, \quad \Lambda^{[i]a} e_{a\nu} p^{\nu} = 0 \,, \quad p_{\mu} p^{\mu} + m^2 = 0 \tag{5}$$

- Solve constraints, supplementary and gauge conditions.
- Find variables, in which Lagrangian is of the canonical form

$$L = p_i \dot{q}^i - H \tag{6}$$



- In detail, the canonical Lagrangian reads: (with $\hat{\Omega}^{(i)(j)} = \hat{\Lambda}^{[k](i)} \dot{\hat{\Lambda}}^{[k](j)}$)

$$L = \frac{1}{16\pi} \int \mathrm{d}^3x \, \hat{\pi}^{ij\mathrm{TT}} \hat{h}^{\mathrm{TT}}_{ij,0} + \hat{p}_i \dot{\hat{z}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} - H^{\mathrm{ADM}} \tag{7}$$

- Then the equations of motion take on the form

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \{A, H^{\mathrm{ADM}}\} + \frac{\partial A}{\partial t} \tag{8}$$

with Poisson brackets

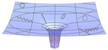
$$\{\hat{z}^i, \hat{p}_j\} = \delta_{ij} \tag{9}$$

$$\{\hat{\Lambda}^{[i](j)}, \hat{S}_{(k)(l)}\} = \hat{\Lambda}^{[i](k)} \delta_{lj} - \hat{\Lambda}^{[i](l)} \delta_{kj} \tag{10}$$

$$\{\hat{S}_{(i)(j)}, \hat{S}_{(k)(l)}\} = \delta_{ik} \hat{S}_{(j)(l)} - \delta_{jk} \hat{S}_{(i)(l)} - \delta_{il} \hat{S}_{(j)(k)} + \delta_{jl} \hat{S}_{(i)(k)} \tag{11}$$

$$\{\hat{h}^{\mathrm{TT}}_{ij}(\mathbf{x}), \hat{\pi}^{kl\mathrm{TT}}(\mathbf{x}')\} = 16\pi \delta^{\mathrm{TT}kl}_{ij} \delta(\mathbf{x} - \mathbf{x}') \tag{12}$$

$$\delta^{\mathrm{TT}ij}_{kl} \equiv \text{TT-projector}$$



Canonical Variables to Linear Order in Spin

- Gauges [cf. Kibble 1963]: $e_{(0)\mu} = n_\mu, \quad (e_{(i)j}) = \sqrt{(\gamma_{ij})}, \quad \tau = t$
- Matter variables: compatible with SSC $\hat{S}^{\mu\nu}(p_\nu + m n_\nu) = 0$

$$z^i = \hat{z}^i - \frac{nS^i}{m - np}, \quad np = -\sqrt{m^2 + \gamma^{ij} p_i p_j}$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i nS_j}{m - np} + \frac{p_j nS_i}{m - np}, \quad nS_i = -\frac{p_k \gamma^{kj} \hat{S}_{ji}}{m}$$

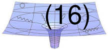
$$\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)} \left(\delta_{kj} + \frac{p_{(k)} p_{(j)}}{m(m - np)} \right), \quad \gamma_{ik}\gamma_{jl} A^{kl} = \frac{1}{2}\hat{S}_{ij} + \frac{m p_{(i} nS_{j)}}{np(m - np)}$$

$$p_i = \hat{p}_i - K_{ij} nS^j - A^{kl} e_{(j)k} e^{(j)}{}_{l,i} + \left(\frac{1}{2} S_{kj} + \frac{p_{(k} nS_{j)}}{np} \right) \Gamma^{kj}{}_i \tag{13}$$

- Field variables:

$$h^{\mathrm{TT}}_{ij} = \hat{h}^{\mathrm{TT}}_{ij} \tag{14}$$

$$\pi^{ij\mathrm{TT}} = \hat{\pi}^{ij\mathrm{TT}} - \delta^{\mathrm{TT}ij}_{kl} (8\pi A^{(kl)} \delta + 16\pi B^{kl}_{mn} A^{[mn]} \delta) \tag{15}$$



$$2B^{kl}_{mn} \equiv e^{(i)}{}_m \frac{\partial e_{(i)n}}{\partial \gamma_{kl}} - e^{(i)}{}_n \frac{\partial e_{(i)m}}{\partial \gamma_{kl}} \tag{16}$$

Outline

1. Three-body interactions with spin:
   NLO Spin-Orbit and NLO Spin(a)-Spin(b) Hamiltonians

   - J. Hartung and J. Steinhoff
     Phys. Rev. D, submitted, arXiv:1011.1179

2. Action approach to canonical formulation of spin in GR

   - J. Steinhoff and G. Schäfer
     Europhys. Lett. **87**, 50004 (2009)

3. Quadrupole deformation of Neutron stars due to Spin:
   NLO Spin(1)-Spin(1) Hamiltonian

   - S. Hergt, J. Steinhoff, and G. Schäfer
     Class. Quant. Grav. **27**, 135007 (2010)

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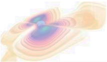
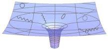




Quadrupole Deformation due to Spin

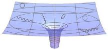
- Quadratic order in spin $\rightarrow$ quadrupole deformation
- Ansatz for Dixon's quadrupole:

$$J^{\nu\rho\beta\alpha} = -3u^{[\nu}Q^{\rho][\beta}u^{\alpha]}\,, \qquad Q_{\mu\nu} = \frac{C_Q}{m_p} S_{\mu\rho} S_{\nu}{}^{\rho} - \text{Trace} \tag{17}$$

- C_Q is an object dependent constant:
  - Neutron stars: $C_Q = 4 \dots 8$ [Poisson (1998); Laarakkers, Poisson (1999)]
  - Black holes: $C_Q = 1$
- Approach via effective action possible [cf. Porto, Rothstein (2008)]:

$$L_{S^2} = \underbrace{-\frac{1}{2m} \mathrm{R}_{\mu\nu\alpha\beta} S^{\rho\mu} S^{\alpha\beta} \frac{u^{\nu} u_{\rho}}{\sqrt{-u_{\sigma} u^{\sigma}}}}_{\text{preserves supplementary conditions}} \underbrace{-\frac{1}{2} \mathrm{R}_{\alpha\mu\beta\nu} Q^{\alpha\beta} \frac{u^{\mu} u^{\nu}}{\sqrt{-u_{\sigma} u^{\sigma}}}}_{\text{quadrupole deformation}} \tag{18}$$

$$J^{\mu\nu\alpha\beta} = -6 \frac{\partial L_{S^2}}{\partial \mathrm{R}_{\mu\nu\alpha\beta}} \qquad \text{[Bailey, Israel (1975)]} \tag{19}$$



- $K_{ij,0}$ in matter action problematic for canonical formulation.

Shortcut to NLO Spin(1)-Spin(1) Hamiltonian

- $\hat{p}_i$-dependent part of $H^{\text{NLO}}_{S_1^2}$ follows from Poincaré Algebra.
  [Hergt, Schäfer (2008)]
- Only need $\hat{p}_i = 0$ part of $H^{\text{NLO}}_{S_1^2}$.
- Only need $\hat{p}_i = 0$ part of matter energy density.
- $\hat{p}_i = 0$ part of matter energy density can be calculated from

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \Big[u^{(\mu} p^{\nu)} \delta_{(4)} + \left(u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha} + \frac{1}{3} R_{\alpha\beta\rho}{}^{(\mu} J^{\nu)\rho\beta\alpha} \delta_{(4)} - \frac{2}{3} \left(J^{\mu\alpha\beta\nu} \delta_{(4)} \right)_{||(\alpha\beta)} \Big] \tag{20}$$

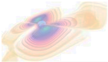
[Steinhoff, Pützfeld (2010)]

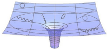
$$J^{\nu\rho\beta\alpha} = -3 u^{[\nu} Q^{\rho][\beta} u^{\alpha]} \,, \qquad Q_{\mu\nu} = \frac{C_Q}{m_p} S_{\mu\rho} S_\nu{}^\rho - \text{Trace} \tag{21}$$

NLO Spin(1)-Spin(1) for Neutron Stars

For black holes: Steinhoff, Hergt, Schäfer (2008). See also: Porto, Rothstein (2008).

$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_1^2} = {}& \frac{m_2}{m_1^3 \hat{r}_{12}^3} \Bigg[\left(\frac{15}{4} - \frac{9}{2} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) + \left(\frac{5}{4} - \frac{5}{4} C_Q \right) \hat{\mathbf{p}}_1^2 \hat{\mathbf{S}}_1^2 \\
& + \left(-\frac{9}{8} + \frac{3}{2} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \hat{\mathbf{S}}_1^2 + \left(-\frac{21}{8} + \frac{9}{4} C_Q \right) \hat{\mathbf{p}}_1^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& + \left(-\frac{5}{4} + \frac{3}{2} C_Q \right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)^2 \Bigg] + \frac{C_Q}{m_1 m_2 \hat{r}_{12}^3} \left[\frac{9}{4} \hat{\mathbf{p}}_2^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 - \frac{3}{4} \hat{\mathbf{p}}_2^2 \hat{\mathbf{S}}_1^2 \right] \\
& + \frac{1}{m_1^2 \hat{r}_{12}^3} \Bigg[\left(-\frac{3}{2} + \frac{9}{2} C_Q \right) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) \\
& + \left(-3 + \frac{3}{2} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) + \left(-\frac{3}{2} + \frac{9}{4} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \hat{\mathbf{S}}_1^2 \\
& + \left(\frac{3}{2} - \frac{3}{4} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \hat{\mathbf{S}}_1^2 + \left(3 - \frac{21}{4} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& - \frac{15}{4} C_Q (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 + \left(\frac{3}{2} - \frac{3}{2} C_Q \right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) \Bigg] \\
& + \frac{m_2}{\hat{r}_{12}^4} \left[\left(2 + \frac{1}{2} C_Q \right) \hat{\mathbf{S}}_1^2 - \left(3 + \frac{3}{2} C_Q \right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \right] \\
& + \frac{m_2^2}{m_1 \hat{r}_{12}^4} \left[(1 + 2 C_Q) \hat{\mathbf{S}}_1^2 - (1 + 6 C_Q)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \right]
\end{aligned}
$$





Thank you for your attention

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