The ADM canonical formalism for gravitating spinning objects

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Outline

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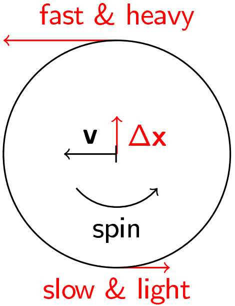
Frame-Dependence of Center and Spin in SR



- Spinning object moving with velocity **v**.
- Shall have constant density in rest frame.
- Upper hemisphere faster than lower.
- Upper hemisphere more massive than lower.
- Center of mass displaced by $\Delta\mathbf{x}$.
- Spin depends on location of center.

- Description by means of a 4-tensor $S^{\mu\nu}$:
  - Spin is $S^{ij} = \epsilon^{ijk} S_k$.
  - Mass dipole related to S^{0i}.
- Spin supplementary condition (SSC) fixates S^{0i} in terms of S^{ij}.

SSC in SR

- Usefull SSCs are, with mass m and 4-momentum p_μ:
  - Møller SSC: $\tilde{S}^{0\mu} = 0$
  - Fokker-Synge-Pryce (covariant) SSC: $S^{\mu\nu} p_\nu = 0$
  - Newton-Wigner (canonical) SSC: $m\hat{S}^{0\mu} - \hat{S}^{\mu\nu} p_\nu = 0$
- Canonical structure depends on SSC, and can be complicated.
- In covariant SSC, with position $\mathbf{z}$:

$$\{z^i(t), z^j(t)\} = \frac{S^{ij}}{m^2} - \frac{p^i S^{0j} - p^j S^{0i}}{m^2 p^0}, \quad \ldots$$

- In Newton-Wigner SSC:

$$\{\hat{z}^i(t), p_j(t)\} = \delta_{ij}, \quad \{\hat{S}_i(t), \hat{S}_j(t)\} = \epsilon_{ijk} \hat{S}_k(t)$$

$$\{\hat{\mathbf{S}}^2, \ldots\} = 0 \quad \Rightarrow \quad \hat{\mathbf{S}}^2 = \text{const.}$$

Spin in GR

- We restrict to linear order in spin here:
  - No deformation by spin included.
  - Linear order is universal.
- Stress-Energy Tensor in covariant SSC:

$$\sqrt{-g}\; T^{\mu\nu} = \int d\tau \left[m u^{\mu} u^{\nu} \delta_{(4)} - (S^{\alpha(\mu} u^{\nu)} \delta_{(4)})_{;\alpha} \right]$$

$$= p^{\mu} v^{\nu} \delta - (S^{\alpha(\mu} v^{\nu)} \delta)_{,\alpha} - S^{\alpha(\mu} \Gamma^{\nu)}_{\alpha\beta} v^{\beta} \delta$$

$$\delta_{(4)} \equiv \delta(x - z)\,, \quad \delta \equiv \delta(\mathbf{x} - \mathbf{z})$$

- EOM follow from $T^{\mu\nu}{}_{;\nu} = 0$:

$$\frac{DS^{\mu\nu}}{d\tau} = 0\,, \quad \frac{Dp_{\mu}}{d\tau} = \frac{1}{2} S^{\lambda\nu} u^{\gamma} R_{\mu\gamma\nu\lambda}$$

- Actions are known for covariant SSC in external grav. fields.

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Some Literature on Spin in Relativity

- G. N. Fleming
  Covariant Position Operators, Spin, and Locality
  Phys. Rev. **137**, B 188 (1965)
- A. J. Hanson and T. Regge
  The Relativistic Spherical Top
  Ann. Phys. (N.Y.) **87**, 498 (1974)
- A. Trautman
  Lectures on General Relativity
  Gen. Rel. Grav. **34**, 721 (2002)
- J. Natário
  Tangent Euler Top in General Relativity
  arXiv:gr-qc/0703081v1

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Preliminaries

- $3+1$ decomposition:

$$g_{\mu\nu} = \begin{pmatrix} N^i N_i - N^2 & N_i \\ N_j & \gamma_{ij} \end{pmatrix}, \quad N^2 g^{\mu\nu} = \begin{pmatrix} -1 & N^i \\ N^j & N^2\gamma^{ij} - N^i N^j \end{pmatrix}$$

$$K_{ij} = -N\Gamma^0_{ij}, \quad n_\mu = (-N, 0, 0, 0)$$

- Canonical Variables:

$$\{\gamma_{ij}(\mathbf{x}, t), \pi^{kl}(\mathbf{x}', t)\} = 16\pi\delta^{kl}_{ij}\delta(\mathbf{x} - \mathbf{x}')$$

- In the following, we will always restrict to:

$$\pi^{ij} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{kl}$$

Hamiltonian before Gauge Fixing

$$H = \int d^3\mathbf{x}(N\mathcal{H} - N^i\mathcal{H}_i) + E[\gamma_{ij}]$$

$$\mathcal{H} = \mathcal{H}^{\mathrm{field}} + \mathcal{H}^{\mathrm{matter}}$$

$$\mathcal{H}_i = \mathcal{H}_i^{\mathrm{field}} + \mathcal{H}_i^{\mathrm{matter}}$$

$$\mathcal{H}^{\mathrm{field}} = -\frac{1}{16\pi\sqrt{\gamma}}\left[\gamma \mathrm{R} + \frac{1}{2}\left(\gamma_{ij}\pi^{ij}\right)^2 - \gamma_{ij}\gamma_{kl}\pi^{ik}\pi^{jl}\right]$$

$$\mathcal{H}_i^{\mathrm{field}} = \frac{1}{8\pi}\gamma_{ij}\pi^{jk}{}_{;k}$$

- Lapse N and shift N^i are Lagrange multipliers.
- Surface term $E[\gamma_{ij}]$ reflects boundary conditions.
- 3-dim. geometry of $t =$ const. surfaces important.

Field Equations before Gauge Fixing

$$H = \int d^3\mathbf{x}(N\mathcal{H} - N^i\mathcal{H}_i) + E[\gamma_{ij}]$$

- 12 evolution equations:

$$\frac{1}{16\pi}\frac{\partial \pi^{ij}}{\partial t} = -\frac{\delta H}{\delta \gamma_{ij}}, \qquad \frac{1}{16\pi}\frac{\partial \gamma_{ij}}{\partial t} = \frac{\delta H}{\delta \pi^{ij}}$$

- Four constraint equations:

$$\frac{\delta H}{\delta N} \equiv \mathcal{H} = 0, \quad -\frac{\delta H}{\delta N^i} \equiv \mathcal{H}_i = 0$$

- Compare with $3+1$ version of the Einstein equations:

$$\mathcal{H}^{\text{matter}} = \sqrt{\gamma}\, T^{\mu\nu} n_\mu n_\nu\,, \qquad \mathcal{H}_i^{\text{matter}} = -\sqrt{\gamma}\, T_i^{\nu} n_\nu$$

- Matter parts can be calculated using the stress-energy tensor, cf. Boulware and Deser (1967).

The ADMTT Gauge

- ADMTT gauge is defined by:

$$0 = 3\gamma_{ij,j} - \gamma_{jj,i}$$
$$0 = \pi^{ii}$$

- Equivalent to a decomposition:

$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$
$$\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}}$$

- Also fixates Lapse and Shift:

$$(3\gamma_{ij,j} - \gamma_{jj,i})_{,0} = 0 \quad \Rightarrow \quad 3\Delta N_i + N_{j,ji} = \ldots$$
$$\pi^{ii}{}_{,0} = 0 \quad \Rightarrow \quad \Delta N = \ldots$$

The Reduced Hamiltonian

- Constraints together with gauge conditions allow reduction of phase space.
- Reduction in ADMTT gauge:

$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$
$$\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}}$$

- Constraints are solved for ϕ and $\tilde{\pi}^{ij}$.
- Remaining canonical field variables:

$$\{h_{ij}^{\mathrm{TT}}(\mathbf{x}, t), \pi^{kl}_{\mathrm{TT}}(\mathbf{x}', t)\} = 16\pi \delta_{ij}^{\mathrm{TT}kl} \delta(\mathbf{x} - \mathbf{x}')$$

- Surface expression E turns into reduced Hamiltonian H_{ADM}.
- For general gauges, Dirac-brackets must be used.

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Global Poincaré Invariance I

- Global Poincaré group is a consequence of asymptotic flatness.
- Generators P^μ and $J^{\mu\nu}$ are conserved.
- Poincaré algebra:

$$
\begin{aligned}
\{P^\mu, P^\nu\} &= 0 \\
\{P^\mu, J^{\rho\sigma}\} &= -\eta^{\mu\rho} P^\sigma + \eta^{\mu\sigma} P^\rho \\
\{J^{\mu\nu}, P^{\rho\sigma}\} &= -\eta^{\nu\rho} J^{\mu\sigma} + \eta^{\mu\rho} J^{\nu\sigma} + \eta^{\sigma\mu} J^{\rho\nu} - \eta^{\sigma\mu} J^{\rho\mu}
\end{aligned}
$$

- $3+1$ decomposition:
  - Energy: $E \equiv P^0$
  - Momentum: P^i
  - Angular momentum: $J^i \equiv \frac{1}{2}\epsilon^{ijk} J_{jk}$
  - Boost: $J^{i0} \equiv K^i \equiv G^i - t\, P^i$
  - Center of mass: $X^i \equiv G^i / E$

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Global Poincaré Invariance II

- E, P^i, J^i, and G^i are given as surface integrals at spatial infinity.
- Poincaré algebra can be checked after reduction.
- E and G^i in ADMTT gauge:

$$E = -\frac{1}{16\pi}\int \mathrm{d}^3\mathbf{x}\,\Delta\phi$$
$$G^i = -\frac{1}{16\pi}\int \mathrm{d}^3\mathbf{x}\,x^i\Delta\phi$$

- P_i and J_{ij} in ADMTT gauge (2PN):

$$P_i = \int d^3\mathbf{x}\,\mathcal{H}_i^{\mathrm{matter}}$$
$$J_{ij} = \int d^3\mathbf{x}\,(x^i\mathcal{H}_j^{\mathrm{matter}} - x^j\mathcal{H}_i^{\mathrm{matter}})$$

Some Literature on the ADM Formalism

R. Arnowitt, S. Deser, and C. W. Misner
The Dynamics of General Relativity
in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962); arXiv:gr-qc/0405109

A. Hanson, T. Regge, and C. Teitelboim
Constrained Hamiltonian Systems
Academia Nazionale dei Lincei, Roma, 1976

P. Jaranowski and G. Schäfer
Third post-Newtonian higher order ADM Hamiltonian dynamics for two-body point-mass systems
Phys. Rev. D **57**, 7274 (1998)

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Conditions on the Matter Hamiltonian

$$H^{\text{matter}} = \int d^3\mathbf{x}(N\mathcal{H}^{\text{matter}} - N^i\mathcal{H}_i^{\text{matter}})$$

- $\mathcal{H}^{\text{matter}}$ and $\mathcal{H}_i^{\text{matter}}$ must be independent of N and N^i.
- Constraints coincide with the Einstein equations iff:

$$\mathcal{H}^{\text{matter}} = \sqrt{\gamma}\, T^{\mu\nu} n_\mu n_\nu \,, \qquad \mathcal{H}_i^{\text{matter}} = -\sqrt{\gamma}\, T_i^{\nu} n_\nu$$

- Evolution equations coincide with the Einstein equations iff:

$$\frac{\delta H^{\text{matter}}}{\delta \pi^{ij}} = 0 \,, \qquad \frac{\delta H^{\text{matter}}}{\delta \gamma^{ij}} = \frac{1}{2} N \sqrt{\gamma}\, T_{ij}$$

- $\mathcal{H}^{\text{matter}}$ and $\mathcal{H}_i^{\text{matter}}$ must be independent of N, N^i and π^{ij}.
- Construct them as 3-dim. covariant generalisations of their Minkowski versions.

Our Strategy

- Calculate matter parts using stress-energy tensor in Minkowski space in covariant SSC:

$$\mathcal{H}^{\text{matter}} = \sqrt{\gamma} T^{\mu\nu} n_\mu n_\nu \,, \qquad \mathcal{H}_i^{\text{matter}} = -\sqrt{\gamma} T_i^{\nu} n_\nu$$

- Go over to Newton-Wigner SSC.
- Take 3-dim. covariant generalisations of $\mathcal{H}^{\text{matter}}$ and $\mathcal{H}_i^{\text{matter}}$.
- Redefine momentum, such that $P_i = \int d^3\mathbf{x}\, \mathcal{H}_i^{\text{matter}}$.
- Redefine spin, such that $\mathbf{S}^2 = \text{const.}$
- Questions that must be answered:
  - Is the calculation with the 4-dim. covariant stress-energy tensor possible?
  - Is $\frac{\delta H^{\text{matter}}}{\delta \gamma^{ij}} = \frac{1}{2} N \sqrt{\gamma} T_{ij}$ fulfilled?
  - Is the Poincaré algebra fulfilled?

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Minkowski Space Versions

- Stress-Energy tensor in covariant SSC:

$$T^{\mu\nu} = p^{\mu} v^{\nu} \delta - (S^{\alpha(\mu} v^{\nu)} \delta)_{,\alpha}$$

- Use Newton-Wigner variables, $np \equiv n_\mu p^\mu = -\sqrt{m^2 + \gamma^{ij} p_i p_j}$:

$$\hat{z}^{\mu} = z^{\mu} - \frac{S^{\mu\nu} n_{\nu}}{m - np}$$

$$S^{\mu\nu} = \hat{S}^{\mu\nu} + p^{\mu} n_{\lambda} \hat{S}^{\nu\lambda}/m - p^{\nu} n_{\lambda} \hat{S}^{\mu\lambda}/m$$

- Result:

$$\sqrt{\gamma}\, \hat{T}^{\mu\nu} n_{\mu} n_{\nu} = -np\delta - \left[\delta_{ij}\delta_{kl} \frac{p_l}{m - np} \hat{S}_{jk} \delta\right]_{,i}$$

$$-\sqrt{\gamma}\, \hat{T}^{\nu}_{i} n_{\nu} = p_i \delta + \frac{1}{2}\left[\delta_{mk} \hat{S}_{ik} \delta\right]_{,m} - \left[(\delta_{mk}\delta_{ip} + \delta_{mp}\delta_{ik})\delta_{ql} \hat{S}_{qp} \frac{p_l p_k}{np(m - np)} \delta\right]_{,m}$$

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Redefinition of Momentum

- 3-dim. covariant generalisation:

$$\mathcal{H}^{\text{matter}} = -np\delta - \left[\gamma^{ij}\gamma^{kl}\frac{p_l}{m-np}\hat{S}_{jk}\delta\right]_{,i}$$

$$\mathcal{H}_i^{\text{matter}} = p_i\delta + \frac{1}{2}\left[\gamma^{mk}\hat{S}_{ik}\delta\right]_{;m} - \left[(\gamma^{mk}\delta_i^p + \gamma^{mp}\delta_i^k)\gamma^{ql}\hat{S}_{qp}\frac{p_l p_k}{np(m-np)}\delta\right]_{;m}$$

- Redefine momentum, such that $P_i = \int d^3\mathbf{x}\,\mathcal{H}_i^{\text{matter}}$:

$$\mathcal{H}_i^{\text{matter}} = P_i\delta + [\dots]_{,m}$$

$$P_i \equiv p_i - \frac{1}{2}\left[\gamma^{lj}\gamma^{kp}\gamma_{il,p} - \frac{p_m p_q}{np(m-np)}\gamma^{mj}\gamma^{kl}\gamma^{qp}\gamma_{lp,i}\right]\hat{S}_{jk}$$

Redefinition of Spin

- In Minkowski space, we have:

$$\hat{S}_{ij}\hat{S}_{ij} = \text{const.}$$

- Covariant generalisation:

$$\gamma^{ik}\gamma^{jl}\hat{S}_{ij}\hat{S}_{kl} = \text{const.}$$

- Need symmetric root of γ_{ij} (dreibein), cf. Kibble (1963):

$$e_{il}e_{lj} = \gamma_{ij}\,, \quad e_{ij} = e_{ji}$$

- Constant-euclidean-length spin $S_{(i)(j)} = \epsilon_{ijk}S_{(k)}$:

$$\hat{S}_{kl} = e_{ki}e_{lj}S_{(i)(j)}, \quad S_{(i)(j)}S_{(i)(j)} = const.$$

The Final Result for our Formalism

$$\mathcal{H}^{\text{matter}} = -nP\delta - \frac{1}{2} t^k_{ij} \gamma^{ij}{}_{,k} - \left[\frac{P_l}{m - nP} \gamma^{ij}\gamma^{kl}\hat{S}_{jk}\delta\right]_{,i}$$

$$\mathcal{H}^{\text{matter}}_i = P_i\delta + \frac{1}{2}\left[\gamma^{mk}\hat{S}_{ik}\delta\right]_{,m} - \left[\frac{P_l P_k}{nP(m-nP)}(\gamma^{mk}\delta^p_i + \gamma^{mp}\delta^k_i)\gamma^{ql}\hat{S}_{qp}\delta\right]_{,m}$$

$$\sqrt{\gamma}\,\hat{T}_{ij} = -\frac{P_i P_j}{nP}\delta + t^k_{ij,k} + \mathcal{O}(G)$$

$$t^k_{ij} \equiv \gamma^{kl}\frac{\hat{S}_{l(i}P_{j)}}{nP}\delta + \gamma^{kl}\gamma^{mn}\frac{\hat{S}_{m(i}P_{j)}P_n P_l}{(nP)^2(m-nP)}\delta$$

Correct evolution equations?

$$H^{\text{matter}} = \int d^3\mathbf{x}(N\mathcal{H}^{\text{matter}} - N^i\mathcal{H}_i^{\text{matter}})$$

- Inspection of our matter expressions:

$$\mathcal{H}^{\text{matter}} = -nP\delta - \frac{1}{2}t_{ij}^{k}\gamma^{ij}{}_{,k} - \left[\frac{P_l}{m-nP}\gamma^{ij}\gamma^{kl}\hat{S}_{jk}\delta\right]_{,i}$$

$$\sqrt{\gamma}\hat{T}_{ij} = -\frac{P_iP_j}{nP}\delta + t_{ij,k}^{k} + \mathcal{O}(G)$$

- We have at least:

$$\frac{\delta H^{\text{matter}}}{\delta\gamma^{ij}} = \frac{1}{2}N\sqrt{\gamma}\hat{T}_{ij} + \mathcal{O}(G)$$

- Sufficient for 2PN $\hat{=}$ NLO!

Global Poincaré Invariance: Preliminary Results

$$\mathcal{H}_i^{\mathrm{matter}} = P_i\delta + \frac{1}{2}\left[\gamma^{mk}\hat{S}_{ik}\delta\right] - \left[\frac{P_l P_k}{nP(m-nP)}(\gamma^{mk}\delta_i^p + \gamma^{mp}\delta_i^k)\gamma^{ql}\hat{S}_{qp}\delta\right]_{,m}$$

- By construction, we have: $P_i = \int d^3\mathbf{x}\,\mathcal{H}_i^{\mathrm{matter}}$
- We also have:

$$J_{ij} = \int d^3\mathbf{x}\,(x^i\mathcal{H}_j^{\mathrm{matter}} - x^j\mathcal{H}_i^{\mathrm{matter}}) = \hat{z}^i P_j - \hat{z}^j P_i + S_{(i)(j)}$$

- Implies that a major part of the Poincaré algebra is fulfilled!
- Justifies the use of standard Poisson-brackets for matter in the ADMTT gauge:

$$\{\hat{z}^i(t), P_j(t)\} = \delta_{ij}\,, \quad \{S_{(i)}(t), S_{(j)}(t)\} = \epsilon_{ijk}S_{(k)}(t)$$

Result with 4-dim. Covariant Stress-Energy Tensor

- Generalisation of Newton-Wigner SSC:

$$\delta x^{\mu} = -\frac{S^{\mu\nu} n_{\nu}}{m - np}$$

$$S^{\mu\nu} = \hat{S}^{\mu\nu} + p^{\mu} n_{\lambda} \hat{S}^{\nu\lambda}/m - p^{\nu} n_{\lambda} \hat{S}^{\mu\lambda}/m$$

- We need an additional term in P_i: $P_i = p_i - n_{\mu} S^{k\mu} K_{ik} + \dots$
- We add a Lie-shift:

$$\sqrt{\gamma} T^{\mu\nu} n_{\mu} n_{\nu} + \mathcal{L}_{m\delta x^{\sigma}} [\sqrt{\gamma} T^{\mu\nu} n_{\mu} n_{\nu}] = \mathcal{H}^{\mathrm{matter}}$$

$$- \sqrt{\gamma} T^{\nu}_{i} n_{\nu} + \mathcal{L}_{m\delta x^{\sigma}} [-\sqrt{\gamma} T^{\nu}_{i} n_{\nu}] = \mathcal{H}^{\mathrm{matter}}_{i} - \delta x^{j} (P_{i;j} + P_{j,i} - P_{i,j}) \delta$$

- P_i must be parallel shifted along δx^j without rotation.
- Agreement with our previous result!

Action Principle

$$W = \int dt \Bigg(\sum_a P_{ai} \dot{\hat{z}}_a^i + \sum_a S_a^{(i)} \Omega_a^{(i)} + \frac{1}{16\pi} \int d^3x \, \pi^{ij}_{\mathrm{TT}} \dot{h}^{\mathrm{TT}}_{ij} - H_{ADM}\left[\hat{z}_a^i, P_{ai}, S_a^{(j)}, h^{\mathrm{TT}}_{ij}, \pi^{ij}_{\mathrm{TT}}\right] \Bigg)$$

- Formulas: $\Omega_a^{(i)} = \frac{1}{2}\epsilon_{ijk}\Lambda_{a(l)(j)}\dot{\Lambda}_{a(l)(k)}$,
  $\Lambda_{a(i)(k)}\Lambda_{a(j)(k)} = \Lambda_{a(k)(i)}\Lambda_{a(k)(j)} = \delta_{ij}$.
- Variables to vary: P_{ai}, $\hat{z}_a^i$, $S_a^{(i)}$, $\Lambda_{a(i)(j)}$.
- Equations of motion for matter:

$$\dot{\hat{z}}_a^i(t) = \frac{\delta \int dt' H_{ADM}}{\delta P_{ai}(t)}, \quad \dot{P}_{ai}(t) = -\frac{\delta \int dt' H_{ADM}}{\delta \hat{z}_a^i(t)}$$

$$\Omega_a^{(i)}(t) = \frac{\delta \int dt' H_{ADM}}{\delta S_a^{(i)}(t)}, \quad \dot{S}_a^{(i)}(t) = \epsilon_{ijk}\Omega_a^{(j)}(t) S_a^{(k)}(t)$$

Outline

1. Introduction
   - Spinning objects in SR and GR
   - The ADM formalism
2. Our Formulation
   - Strategy of our approach
   - Details on the derivation
   - Gauge independent formalism?
3. Application
   - Hamiltonians
   - Comparison with other Methods

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The Constraint Algebra I

$$\{\mathcal{H}(x), \mathcal{H}(x')\} = -\left[\mathcal{H}_i(x)\gamma^{ij}(x) + \mathcal{H}_i(x')\gamma^{ij}(x')\right] \delta_{\mathbf{xx}',j}$$
$$\{\mathcal{H}_i(x), \mathcal{H}(x')\} = -\mathcal{H}(x)\,\delta_{\mathbf{xx}',i}$$
$$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = -\mathcal{H}_j(x)\,\delta_{\mathbf{xx}',i} - \mathcal{H}_i(x')\,\delta_{\mathbf{xx}',j}$$

- This algebra is valid for point-masses.
- The coupling to gravity must be simple:
  - $\mathcal{H}^{\mathrm{matter}}$ does not depend on derivatives of γ_{ij}.
  - $\mathcal{H}_i^{\mathrm{matter}}$ does not depend on γ_{ij} at all.
- Spinning objects do not couple that simple.
- An algebra of (first-class) constraints is related to gauge symmetries.
- Algebra quite robust even if matter is coupled to gravity.
- Extension of this algebra is possible, if gauge structure is extended.

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The Constraint Algebra II

- Consistent first-class constraint algebra necessary for gauge independent formulation.
- Mixed matter-field contribution for simple coupling:

$$\{\mathcal{H}_i^{\mathrm{field}}(x), \mathcal{H}^{\mathrm{matter}}(x')\} = \sqrt{\gamma}\, T_{jk}(x') \left[\delta_i^j \gamma^{kl}(x')\, \delta_{\mathbf{xx'},l} + \gamma^{jk}{}_{,i}(x')\, \delta_{\mathbf{xx'}}\right]$$

- Matter-only algebra for simple coupling:

$$\{\mathcal{H}^{\mathrm{m}}(x), \mathcal{H}^{\mathrm{m}}(x')\} = -\left[\mathcal{H}_i^{\mathrm{m}}(x)\gamma^{ij}(x) + \mathcal{H}_i^{\mathrm{m}}(x')\gamma^{ij}(x')\right] \delta_{\mathbf{xx'},j}$$

$$\{\mathcal{H}_i^{\mathrm{m}}(x), \mathcal{H}^{\mathrm{m}}(x')\} = -\mathcal{H}^{\mathrm{m}}(x)\, \delta_{\mathbf{xx'},i} - \sqrt{\gamma}\, T_{jk}(x') \left[\delta_i^j \gamma^{kl}(x')\, \delta_{\mathbf{xx'},l} + \gamma^{jk}{}_{,i}(x')\, \delta_{\mathbf{xx'}}\right]$$

$$\{\mathcal{H}_i^{\mathrm{m}}(x), \mathcal{H}_j^{\mathrm{m}}(x')\} = -\mathcal{H}_j^{\mathrm{m}}(x)\, \delta_{\mathbf{xx'},i} - \mathcal{H}_i^{\mathrm{m}}(x')\, \delta_{\mathbf{xx'},j}$$

- Minkowski-limit of this algebra can be considered . . .

Algebra of Spinning Objects Stress-Energy-Tensor Components in Minkowski Space

$$\{\mathcal{H}^{\mathrm{m}}(x), \mathcal{H}^{\mathrm{m}}(x')\} = -\left[\mathcal{H}^{\mathrm{m}}_i(x) + \mathcal{H}^{\mathrm{m}}_i(x')\right] \delta_{\mathbf{xx}',i}$$
$$\{\mathcal{H}^{\mathrm{m}}_i(x), \mathcal{H}^{\mathrm{m}}(x')\} = -\mathcal{H}^{\mathrm{m}}(x)\, \delta_{\mathbf{xx}',i} - T_{ij}(x')\, \delta_{\mathbf{xx}',j}$$
$$\{\mathcal{H}^{\mathrm{m}}_i(x), \mathcal{H}^{\mathrm{m}}_j(x')\} = -\mathcal{H}^{\mathrm{m}}_j(x)\, \delta_{\mathbf{xx}',i} - \mathcal{H}^{\mathrm{m}}_i(x')\, \delta_{\mathbf{xx}',j} + \partial_n \partial'_q \left[h_{injq}(x)\, \delta_{\mathbf{xx}'}\right]$$

$$h_{injq}(x) = \left[- \hat{S}_{q)(n}\mathcal{P}_{i)(j} - \delta^{kl} \frac{p_k \hat{S}_{l(n}\mathcal{P}_{i)(j}p_{q)}}{(np)(m-np)} + \delta^{kl} \frac{p_k \hat{S}_{l(q}\mathcal{P}_{j)(i}p_{n)}}{(np)(m-np)} \right] \delta$$
$$\mathcal{P}_{ij} \equiv \delta_{ij} - \frac{p_i p_j}{(np)^2}$$

- This is a part of the Stress-Energy-Tensor algebra, $\mathcal{H}^{\mathrm{m}}(x) = T^{00}$ and $\mathcal{H}^{\mathrm{m}}_i(x) = T^{0i}$.
- Occurance of $h_{injq}(x)$ shows already in the Minkowsky case that coupling to gravity can not be simple.
- Dirac field also has $h_{injq}(x) \neq 0$.

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Perturbative Solution of Partial Differential Equations

Example:

- $\Delta f(\mathbf{x}) = a(\mathbf{x}) + b(\mathbf{x}) f(\mathbf{x}) + c(\mathbf{x})[f(\mathbf{x})]^2 + \dots$
- In general: vectors, other diff. op., derivatives on RHS, ...
- Perturbative expansion, e.g. $a = a_{(1)} + a_{(2)} + a_{(3)} + \dots$
- Leads to recursive equations:

$$\begin{aligned}
\Delta f_{(1)}(\mathbf{x}) &= a_{(1)}(\mathbf{x}) \\
\Delta f_{(2)}(\mathbf{x}) &= a_{(2)}(\mathbf{x}) + b_{(1)}(\mathbf{x}) f_{(1)}(\mathbf{x}) \\
\Delta f_{(3)}(\mathbf{x}) &= a_{(3)}(\mathbf{x}) + b_{(1)}(\mathbf{x}) f_{(2)}(\mathbf{x}) + b_{(2)}(\mathbf{x}) f_{(1)}(\mathbf{x}) \\
&\quad + c_{(1)}(\mathbf{x})[f_{(1)}(\mathbf{x})]^2 \\
&\ \vdots
\end{aligned}$$

- Delta sources $\Rightarrow$ Regularisation

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Calculation of Hamiltonians

- Going over to ADMTT gauge:

$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$
$$\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}}$$

- Expansion of the constraints in c^{-2}.
- Solve constraints for ϕ and $\tilde{\pi}^{ij}$.
- Calculate Hamiltonian:

$$H_{\mathrm{ADM}}[x_a^i, P_{ai}, S_{a(i)}, h_{ij}^{\mathrm{TT}}, \pi^{kl}_{\mathrm{TT}}] = -\frac{1}{16\pi}\int \mathrm{d}^3\mathbf{x}\,\Delta\phi$$

- Near-zone expansion of wave equation $\Box h_{ij}^{\mathrm{TT}} = \dots$
- Elimination of h_{ij}^{TT} and π^{kl}_{TT}.

The Leading-Order (LO) in Spin

- LO Spin-Orbit Hamiltonian:

$$H^{\rm LO}_{\rm SO} = \sum_a \sum_{b\neq a} \frac{1}{r_{ab}^2} (\mathbf{S}_a \times \mathbf{n}_{ab}) \cdot \left[\frac{3m_b}{2m_a}\mathbf{P}_a - 2\mathbf{P}_b\right]$$

- LO Spin_1-Spin_2 Hamiltonian:

$$H^{\rm LO}_{\rm SS} = \sum_a \sum_{b\neq a} \frac{1}{2r_{ab}^3} [3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b)]$$

- Center of mass vector:

$$\mathbf{G}^{\rm LO}_{\rm SO} = \sum_a \frac{1}{2m_a} (\mathbf{P}_a \times \mathbf{S}_a)\,, \qquad \mathbf{G}^{\rm LO}_{\rm SS} = 0$$

NLO Spin-Orbit Hamiltonian (DJS 2007)

$$
\begin{aligned}
H_{SO}^{NLO} = &-\frac{((\mathbf{P}_1 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^2}\left[\frac{5m_2\mathbf{P}_1^2}{8m_1^3} + \frac{3(\mathbf{P}_1\cdot\mathbf{P}_2)}{4m_1^2} - \frac{3\mathbf{P}_2^2}{4m_1 m_2}\right.\\
&\left. + \frac{3(\mathbf{P}_1\cdot\mathbf{n}_{12})(\mathbf{P}_2\cdot\mathbf{n}_{12})}{4m_1^2} + \frac{3(\mathbf{P}_2\cdot\mathbf{n}_{12})^2}{2m_1 m_2}\right]\\
&+ \frac{((\mathbf{P}_2 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^2}\left[\frac{(\mathbf{P}_1\cdot\mathbf{P}_2)}{m_1 m_2} + \frac{3(\mathbf{P}_1\cdot\mathbf{n}_{12})(\mathbf{P}_2\cdot\mathbf{n}_{12})}{m_1 m_2}\right]\\
&+ \frac{((\mathbf{P}_1 \times \mathbf{S}_1)\cdot \mathbf{P}_2)}{r_{12}^2}\left[\frac{2(\mathbf{P}_2\cdot\mathbf{n}_{12})}{m_1 m_2} - \frac{3(\mathbf{P}_1\cdot\mathbf{n}_{12})}{4m_1^2}\right]\\
&- \frac{((\mathbf{P}_1 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^3}\left[\frac{11m_2}{2} + \frac{5m_2^2}{m_1}\right]\\
&+ \frac{((\mathbf{P}_2 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{ab}^3}\left[6m_1 + \frac{15m_2}{2}\right] + (1 \leftrightarrow 2)
\end{aligned}
$$

NLO Spin$_1$-Spin$_2$ Hamiltonian I

$$
\begin{aligned}
H^{\rm NLO}_{\rm SS} = \frac{1}{2 m_1 m_2 r_{12}^3} [& \tfrac{3}{2}((\mathbf{P}_1 \times \mathbf{S}_1) \cdot \mathbf{n}_{12})((\mathbf{P}_2 \times \mathbf{S}_2) \cdot \mathbf{n}_{12}) \\
& + 6((\mathbf{P}_2 \times \mathbf{S}_1) \cdot \mathbf{n}_{12})((\mathbf{P}_1 \times \mathbf{S}_2) \cdot \mathbf{n}_{12}) \\
& - 15(\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{S}_2 \cdot \mathbf{n}_{12})(\mathbf{P}_1 \cdot \mathbf{n}_{12})(\mathbf{P}_2 \cdot \mathbf{n}_{12}) \\
& - 3(\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{S}_2 \cdot \mathbf{n}_{12})(\mathbf{P}_1 \cdot \mathbf{P}_2) \\
& + 3(\mathbf{S}_1 \cdot \mathbf{P}_2)(\mathbf{S}_2 \cdot \mathbf{n}_{12})(\mathbf{P}_1 \cdot \mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2 \cdot \mathbf{P}_1)(\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{P}_2 \cdot \mathbf{n}_{12}) \\
& + 3(\mathbf{S}_1 \cdot \mathbf{P}_1)(\mathbf{S}_2 \cdot \mathbf{n}_{12})(\mathbf{P}_2 \cdot \mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2 \cdot \mathbf{P}_2)(\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{P}_1 \cdot \mathbf{n}_{12}) \\
& - 3(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{P}_1 \cdot \mathbf{n}_{12})(\mathbf{P}_2 \cdot \mathbf{n}_{12}) + (\mathbf{S}_1 \cdot \mathbf{P}_1)(\mathbf{S}_2 \cdot \mathbf{P}_2) \\
& - \tfrac{1}{2}(\mathbf{S}_1 \cdot \mathbf{P}_2)(\mathbf{S}_2 \cdot \mathbf{P}_1) + \tfrac{1}{2}(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{P}_1 \cdot \mathbf{P}_2)]
\end{aligned}
$$

NLO Spin$_1$-Spin$_2$ Hamiltonian II

$$
\begin{aligned}
&+\frac{3}{2m_1^2 r_{12}^3}[-((\mathbf{P}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{P}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12})\\
&\qquad+(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{P}_1\cdot\mathbf{n}_{12})^2-(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{P}_1)(\mathbf{P}_1\cdot\mathbf{n}_{12})]\\
&+\frac{3}{2m_2^2 r_{12}^3}[-((\mathbf{P}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12})((\mathbf{P}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})\\
&\qquad+(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{P}_2\cdot\mathbf{n}_{12})^2-(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{P}_2)(\mathbf{P}_2\cdot\mathbf{n}_{12})]\\
&+\frac{6(m_1+m_2)}{r_{12}^4}[(\mathbf{S}_1\cdot\mathbf{S}_2)-2(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})]
\end{aligned}
$$

NLO Center of Mass

$$\mathbf{G}^{\mathrm{NLO}}_{\mathrm{SO}} = -\sum_a \frac{\mathbf{P}_a^2}{8m_a^3}(\mathbf{P}_a \times \mathbf{S}_a)$$
$$+ \sum_a \sum_{b\neq a} \frac{m_b}{4m_a r_{ab}}\left[((\mathbf{P}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab})\frac{5\mathbf{x}_a + \mathbf{x}_b}{r_{ab}} - 5(\mathbf{P}_a \times \mathbf{S}_a)\right]$$
$$+ \sum_a \sum_{b\neq a} \frac{1}{r_{ab}}\left[\frac{3}{2}(\mathbf{P}_b \times \mathbf{S}_a) - \frac{1}{2}(\mathbf{n}_{ab}\times \mathbf{S}_a)(\mathbf{P}_b \cdot \mathbf{n}_{ab}) - ((\mathbf{P}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab})\frac{\mathbf{x}_a + \mathbf{x}_b}{r_{ab}}\right]$$

$$\mathbf{G}^{\mathrm{NLO}}_{\mathrm{SS}} = \frac{1}{2}\sum_a \sum_{b\neq a}\left\{\left[3(\mathbf{S}_a\cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b)\right]\frac{\mathbf{x}_a}{r_{ab}^3} + (\mathbf{S}_b\cdot \mathbf{n}_{ab})\frac{\mathbf{S}_a}{r_{ab}^2}\right\}$$

Global Poincaré Invariance

- Check of Poincaré algebra is important:
  - Generally a good indicator for errors.
  - Validates Poisson-brackets for our variables.
- Generators of the Poincaré group:

$$\mathbf{P} = \sum_a \mathbf{P}_a$$

$$\mathbf{J} = \sum_a \mathbf{x}_a \times \mathbf{P}_a + \sum_a \mathbf{S}_a$$

$$\mathbf{G} = \mathbf{G}_{\mathrm{PM}} + \mathbf{G}^{\mathrm{LO}}_{\mathrm{SO}} + \mathbf{G}^{\mathrm{LO}}_{\mathrm{SS}} + \mathbf{G}^{\mathrm{NLO}}_{\mathrm{SO}} + \mathbf{G}^{\mathrm{NLO}}_{\mathrm{SS}}$$

$$H_{\mathrm{ADM}} = H_{\mathrm{PM}} + H^{\mathrm{LO}}_{\mathrm{SO}} + H^{\mathrm{LO}}_{\mathrm{SS}} + H^{\mathrm{NLO}}_{\mathrm{SO}} + H^{\mathrm{NLO}}_{\mathrm{SS}}$$

- Point-mass (PM) contributions must be included.
- Poincaré algebra is fulfilled!

NLO Spin Hamiltonians in the Literature

T. Damour, P. Jaranowski, and G. Schäfer
Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin-orbit coupling
Phys. Rev. D submitted, arXiv:0711.1048v1 [gr-qc]

S. Hergt and G. Schäfer
Source terms for Kerr geometry in approximate ADM coordinates and higher-order-in-spin interaction Hamiltonians for binary black holes
Phys. Rev. D submitted, arXiv:0712.1515v1 [gr-qc]

J. Steinhoff, S. Hergt, and G. Schäfer
On the next-to-leading order gravitational spin(1)-spin(2) dynamics
Phys. Rev. D (R) submitted, arXiv:0712.1716v1 [gr-qc]

Outline

1. Introduction
   - Spinning objects in SR and GR
   - The ADM formalism
2. Our Formulation
   - Strategy of our approach
   - Details on the derivation
   - Gauge independent formalism?
3. Application
   - Hamiltonians
   - Comparison with other Methods

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SO Hamiltonian of Damour, Jaranowski, and Schäfer

arXiv:0711.1048v1 [gr-qc], submitted to Phys. Rev. D

- Hamiltonian is linear in a constant-euclidean-length $\mathbf{S}_a$:

$$H_{\text{SO}}^{\text{NLO}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \sum_{a=1,2} \boldsymbol{\Omega}_a(\mathbf{x}_a, \mathbf{p}_a) \cdot \mathbf{S}_a$$

- EOM from Hamiltonian: $\dot{\mathbf{S}}_a = \boldsymbol{\Omega}_a \times \mathbf{S}_a$
- EOM in covariant SSC: $\dfrac{DS^{\mu\nu}}{d\tau} = 0$
- Compare both EOM using $\mathbf{S}_a^2 = \text{const.}$ (not unique).
- Resulting formula for $\boldsymbol{\Omega}_a$ depends on metric at $\mathbf{x}_a$.
- Metric of point-masses suffices for $\boldsymbol{\Omega}_a$!
- Identical to our result.

Spin_1-Spin_2 Hamiltonian via modified DJS Method

- Ansatz for the Hamiltonian:

$$H_{\text{SS}}^{\text{NLO}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \tilde{\Omega}_{ij}(\mathbf{x}_a, \mathbf{p}_a)\, S_1^{(i)}\, S_2^{(j)} = \mathbf{\Omega}_1(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_2) \cdot \mathbf{S}_1$$

- DJS formula can be used for $\Omega_1^i \equiv \tilde{\Omega}_{ij} S_2^{(j)}$.
- Now $\mathbf{S}_2$-dependent part of the metric is needed.
- Metric is calculated with our source.
- Again identical to our result.
- Lapse and shift were used.
- EOM were used.

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Spin_1-Spin_2 Potential of Porto and Rothstein

- First (incomplete) result:

R. A. Porto and I. Z. Rothstein
Calculation of the First Nonlinear Contribution to the General-Relativistic Spin-Spin Interaction for Binary Systems
Phys. Rev. Lett. **97**, 021101 (2006)

- Prompt confirmation of our spin_1-spin_2 Hamiltonian in arXiv:0712.1716v1:

R. A. Porto and I. Z. Rothstein
Comment on 'On the next-to-leading order gravitational spin(1)-spin(2) dynamics' by J. Steinhoff et al
arXiv:0712.2032v1 [gr-qc]

- Their new, complete potential relates to our Hamiltonian via Legendre and canonical transformation.

Summary

- Our formalism describes the correct NLO spin dynamics.
- Application is formally possible up to any desired order.

- Outlook
  - Further application of our formalism.
  - Gauge independent formulation?
  - Up to which order is our formalism correct?