Canonical formulation of spinning objects in General Relativity

Jan Steinhoff   Steven Hergt   Gerhard Schäfer

since 1558

Theoretisch-Physikalisches Institut
Friedrich-Schiller-Universität Jena

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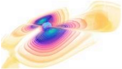

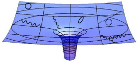

DFG: SFB/TR7 “Gravitational Wave Astronomy” and GRK 1523

Outline

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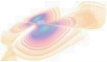

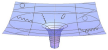

(3+1)-Decomposition

- Normal vector:

$$n_\mu = (-N, 0, 0, 0)\,, \quad n^\mu = \frac{1}{N}(1, -N^i)\,, \quad n_\mu n^\mu = -1$$

- Projector:

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}\,, \quad g_{ij} = \gamma_{ij}\,, \quad \gamma_{ik}\gamma^{kj} = \delta_{ij}$$

- Extrinsic curvature:

$$K_{ij} \equiv -n_{(i||j)}$$

$$\pi^{ij} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{lj} - \gamma^{ij}\gamma^{kl})K_{kl}$$

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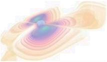
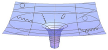




Point-Mass Action in ADM Form

$$W = \int dt\, p_i \dot{z}^i + \int d^4x \left[\frac{1}{16\pi} \pi^{ij} \gamma_{ij,0} - N\mathscr{H} + N^i \mathscr{H}_i + (\mathrm{st}) \right]$$

- Constraint equations:

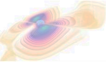
$$0 = \mathscr{H} \equiv -\frac{1}{16\pi\sqrt{\gamma}} \left[\gamma \mathrm{R} + \frac{1}{2} \left(\gamma_{ij} \pi^{ij} \right)^2 - \gamma_{ij} \gamma_{kl} \pi^{ik} \pi^{jl} \right] + \mathscr{H}^{\mathrm{matter}}$$

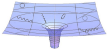
$$0 = \mathscr{H}_i \equiv \frac{1}{8\pi} \gamma_{ij} \pi^{jk}{}_{;k} + \mathscr{H}_i^{\mathrm{matter}}$$

- Source terms are related to the stress-energy tensor $T^{\mu\nu}$:

$$\mathscr{H}^{\mathrm{matter}} \equiv \sqrt{\gamma} T_{\mu\nu} n^\mu n^\nu = \sqrt{m^2 + \gamma^{ij} p_i p_j}\, \delta$$

$$\mathscr{H}_i^{\mathrm{matter}} \equiv -\sqrt{\gamma} T_{i\nu} n^\nu = p_i \delta$$





ADM Canonical Formalism

- Hamiltonian without gauge fixing:

$$H[x_a^i, p_{ai}, \gamma_{ij}, \pi^{ij}] = \int d^3\mathbf{x} (N\mathscr{H} - N^i \mathscr{H}_i) + E[\gamma_{ij}]$$

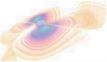
$$E[\gamma_{ij}] = \frac{1}{16\pi} \oint d^2 s_i (\gamma_{ij,j} - \gamma_{jj,i})$$

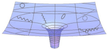
- ADM Hamiltonian (Hamiltonian in ADMTT gauge) $\hat{=}$ ADM Energy depending on canonical variables:

$$H_{\mathrm{ADM}} = E[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}, \pi_{\mathrm{TT}}^{ij}] = -\frac{1}{16\pi} \int \mathrm{d}^3\mathbf{x} \, \Delta\phi$$

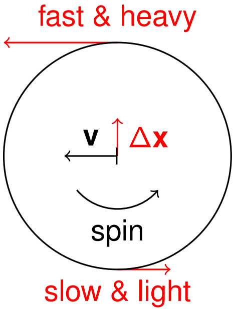
$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}, \quad \pi^{ii} = 0$$

- Matter only Hamiltonian: Elimination of h_{ij}^{TT} and π_{TT}^{ij}.





Spin in Special Relativity



- Different mass centers.
- Spin is a 4-tensor $S^{\mu\nu}$:
  - Spin is $S^{ij} = \varepsilon^{ijk} S_k$.
  - Mass dipole related to S^{i0}.
- Need spin supplementary condition:
  - Møller SSC: $\tilde{S}^{\mu 0} = 0$
  - Covariant SSC: $S^{\mu\nu} p_\nu = 0$
  - Newton-Wigner (canonical) SSC: $m\hat{S}^{\mu 0} + \hat{S}^{\mu\nu} p_\nu = 0$
- In covariant SSC, with position **z**:

$$\{z^i(t), z^j(t)\} = \frac{S^{ij}}{m^2} - \frac{p^i S^{0j} - p^j S^{0i}}{m^2 p^0}, \quad \ldots$$

- In Newton-Wigner SSC:

$$\{\hat{z}^i(t), p_j(t)\} = \delta_{ij}, \quad \{\hat{S}_i(t), \hat{S}_j(t)\} = \varepsilon_{ijk} \hat{S}_k(t)$$

Outline

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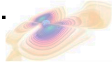
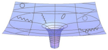
Non-Relativistic Spherical Top

- Body fixed coordinates $\tilde{x}_a^i$: $x_a^i(t) = R_{ji}(t)\tilde{x}_a^j$, $R_{ki}R_{kj} = \delta_{ij}$
- Independent angle variables: $R_{ij} = R_{ij}(\phi, \psi, \theta)$
- Angular velocity tensor: $\Omega^{ij} = \varepsilon_{ijk}\omega^k = R_{ki}\dot{R}_{kj}$
- Lagrangian: $L = \frac{1}{4}J\Omega^{ij}\Omega^{ij}$
- Spin tensor: $S_{ij} = 2\dfrac{\partial L}{\partial \Omega^{ij}} = J\Omega^{ij}$
- Legendre transformed:

$$L = \frac{1}{2}S_{ij}\Omega^{ij} - H[R_{ij}, S_{ij}]\,, \quad H = \frac{1}{4J}S_{ij}S_{ij}$$

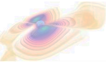
- Trick: Use $\delta\theta_{ij} = -\delta\theta_{ji} = R_{ki}\delta R_{kj}$ as independent variations.
- Usual Poisson brackets.

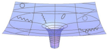




Relativistic Spherical Top

- No rigid bodies.
- Mathematical abstraction: Top is
  - Worldline with Lorentz-matrix $\Lambda_{A\mu}$. $\eta^{AB}\Lambda_{A\mu}\Lambda_{B\nu} = \eta_{\mu\nu}$
  - $\Lambda_{A\mu}$ is pure rotation in rest-frame: $\Lambda_{A\mu} = \begin{pmatrix} -1 & 0 \\ 0 & R_{ij} \end{pmatrix}$
- Equivalent description: $\Lambda_{[0]\mu} = p_\mu / m$ or $\Lambda^{[i]\mu} p_\mu = 0$
- Angular velocity tensor: $\Omega^{\mu\nu} = \Lambda_A{}^\mu \frac{d\Lambda^{A\nu}}{d\tau}$
- Spin tensor: $S_{\mu\nu} = 2\frac{\partial L}{\partial \Omega^{\mu\nu}}$
- Associated SSC: $S_{\mu\nu}p^\mu = 0$





Minimal Coupling

- Problem with metric variation: (also $\Lambda_{A\mu} = e_{A\mu}$)

$$\Lambda_{A\mu}\Lambda^{A}{}_{\nu} = g_{\mu\nu} \quad \leftrightarrow \quad [\gamma_\mu, \gamma_\nu]_A = 2g_{\mu\nu}$$

- Variate Λ^{Aa} and tetrad $e_{a\mu}$, $\Lambda^{A}{}_{\mu} = \Lambda^{Aa} e_{a\mu}$:

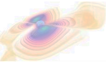
$$\Lambda_{Aa}\Lambda^{A}{}_{b} = \eta_{ab} \quad \leftrightarrow \quad [\gamma_a, \gamma_b]_A = 2\eta_{ab}$$

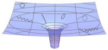
- Matter Lagrangian density and constraints:

$$\mathscr{L}_M = \int d\tau \left[p_\mu u^\mu + \frac{1}{2} S_{ab} \Omega^{ab} \right] \delta_{(4)}$$

$$\Omega^{ab} = \Lambda_A{}^a \frac{d\Lambda^{Ab}}{d\tau} - \omega_\mu{}^{ab} u^\mu$$

$$S_{ab} p^b = 0 \,, \quad \Lambda^{[i]a} p_a = 0 \,, \quad p_\mu p^\mu + m^2 = 0$$





Result of Variation

- Approximated linear in spin.
- Field equations with stress-energy tensor (Mathisson, Tulczyjew):

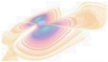
$$\sqrt{-g}\, T^{\mu\nu} = \int d\tau \left[m u^{\mu} u^{\nu} \delta_{(4)} - (S^{\alpha(\mu} u^{\nu)} \delta_{(4)})_{||\alpha} \right]$$

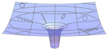
- EOM (Mathisson, Papapetrou):

$$\frac{DS^{\mu\nu}}{d\tau} = 0\,, \qquad \frac{Dp_{\mu}}{d\tau} = \frac{1}{2} S^{\lambda\nu} u^{\gamma} R_{\mu\gamma\nu\lambda}$$

- Derivation: Evaluate $T^{\mu\nu}{}_{||\nu} = 0$ with ansatz

$$\sqrt{-g}\, T^{\mu\nu} = \int d\tau \left[t^{\mu\nu} \delta_{(4)} + (t^{\mu\nu\alpha} \delta_{(4)})_{||\alpha} \right]$$





Reduction of the Matter Part

- Solve matter constraints, Schwinger time gauge $e_{(0)\mu} = n_\mu$, $\tau = t$.
- Variable redefinitions:

$$z^i = \hat{z}^i - \frac{nS^i}{m - np}, \quad np = -\sqrt{m^2 + \gamma^{ij} p_i p_j}$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i nS_j}{m - np} + \frac{p_j nS_i}{m - np}, \quad nS_i = -\frac{p_k \gamma^{kj} \hat{S}_{ji}}{m}$$

$$\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)} \left(\delta_{kj} + \frac{p_{(k)} p^{(j)}}{m(m - np)} \right), \quad \gamma_{ik} \gamma_{jl} A^{kl} = \frac{1}{2} \hat{S}_{ij} + \frac{m p_{(i} nS_{j)}}{np(m - np)}$$

$$\hat{p}_i = p_i + K_{ij} nS^j + A^{kl} e_{(j)k} e^{(j)}_{l,i} - \left(\frac{1}{2} S_{kj} + \frac{p_{(k} nS_{j)}}{np} \right) \Gamma^{kj}{}_i$$

- Matter Lagrangian density now, with $\hat{\Omega}^{(i)(j)} = \hat{\Lambda}_{[k]}{}^{(i)} \dot{\hat{\Lambda}}^{[k](j)}$,

$$\mathscr{L}_M = A^{ij} e_{(k)i} e^{(k)}_{j,0} \hat{\delta} + \hat{p}_i \dot{\hat{z}}^i \hat{\delta} + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} \hat{\delta} - N \mathscr{H}^{\text{matter}} + N^i \mathscr{H}_i^{\text{matter}}$$

ADM Formalism with Spin

- Legendre transformation for gravitational field.
- Spatial symmetric gauge (Kibble 1963): $e_{(i)j} = e_{ij} = e_{ji}$

$$e_{ij} e_{jk} = \gamma_{ik} \quad \Rightarrow \quad (e_{ij}) = \sqrt{(\gamma_{ij})}$$

- Action:

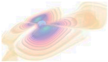
$$W = \frac{1}{16\pi} \int d^4 x \hat{\pi}^{ij} \gamma_{ij,0} + \int dt \left[\hat{p}_i \dot{\hat{z}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} - H \right]$$

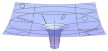
$$H = \int d^3 x (N \mathscr{H} - N^i \mathscr{H}_i) + E[\gamma_{ij}]$$

- Canonical field momentum:

$$\hat{\pi}^{ij} = \pi^{ij} + 8\pi A^{(ij)} \hat{\delta} + 16\pi B^{ij}_{kl} A^{[kl]} \hat{\delta}$$

$$e_{k[i} e_{j]k,0} = B^{kl}_{ij} \gamma_{kl,0}$$





Full Reduction

- Solve field constraints in ADMTT gauge.
- Fully reduced action:

$$W = \frac{1}{16\pi} \int d^4x \, \hat{\pi}^{ij}_{\mathrm{TT}} h^{\mathrm{TT}}_{ij,0} + \int dt \left[\hat{p}_i \dot{\hat{z}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} - H_{\mathrm{ADM}} \right]$$

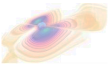
$$H_{\mathrm{ADM}} = E[\hat{z}^i, \hat{p}_i, \hat{S}_{(i)(j)}, h^{\mathrm{TT}}_{ij}, \hat{\pi}^{ij\mathrm{TT}}]$$

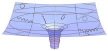
- Fundamental equal-time Poisson brackets:

$$\{\hat{z}^i_a, \hat{p}_{aj}\} = \delta_{ij}, \quad \{\hat{S}_{a(i)}, \hat{S}_{a(j)}\} = \varepsilon_{ijk} \hat{S}_{a(k)}$$

$$\{h^{\mathrm{TT}}_{ij}(\mathbf{x}, t), \hat{\pi}^{kl}_{\mathrm{TT}}(\mathbf{x}', t)\} = 16\pi \delta^{\mathrm{TT}kl}_{ij} \delta(\mathbf{x} - \mathbf{x}')$$

- Valid to all orders linear in spin.





Conserved Quantities

- Energy: $E = H_{\text{ADM}}$
- Total linear and angular momentum:

$$P_i = \sum_a \hat{p}_{ai} - \frac{1}{16\pi} \int d^3x \, \hat{\pi}^{kl}_{\text{TT}} h^{\text{TT}}_{kl,i}$$

$$J_{ij} = \sum_a (\hat{z}^i_a \hat{p}_{aj} - \hat{z}^j_a \hat{p}_{ai}) + \sum_a \hat{S}_{a(i)(j)}$$
$$- \frac{1}{16\pi} \int d^3x \, (x^i \hat{\pi}^{kl}_{\text{TT}} h^{\text{TT}}_{kl,j} - x^j \hat{\pi}^{kl}_{\text{TT}} h^{\text{TT}}_{kl,i})$$
$$- \frac{1}{16\pi} \int d^3x \, 2(\hat{\pi}^{ik}_{\text{TT}} h^{\text{TT}}_{kj} - \hat{\pi}^{jk}_{\text{TT}} h^{\text{TT}}_{ki})$$

- Boost: $J^{i0} \equiv K^i \equiv G^i - t\, P^i$
  With center-of-mass vector:

$$G^i = -\frac{1}{16\pi} \int \mathrm{d}^3\mathbf{x} \, x^i \Delta\phi$$

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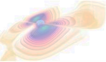
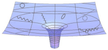




Stress-Energy-Tensor Algebra (Minkowski)

$$\{\mathscr{H}^{\mathrm{m}}(x), \mathscr{H}^{\mathrm{m}}(x')\} = -\mathscr{H}_i^{\mathrm{m}}(x)\delta_{\mathbf{xx'},i} - \mathscr{H}_i^{\mathrm{m}}(x')\delta_{\mathbf{xx'},i}$$

$$\{\mathscr{H}_i^{\mathrm{m}}(x), \mathscr{H}^{\mathrm{m}}(x')\} = -\mathscr{H}^{\mathrm{m}}(x)\,\delta_{\mathbf{xx'},i} - T_{ij}(x')\,\delta_{\mathbf{xx'},j}$$

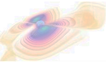
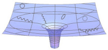
$$\{\mathscr{H}_i^{\mathrm{m}}(x), \mathscr{H}_j^{\mathrm{m}}(x')\} = -\mathscr{H}_j^{\mathrm{m}}(x)\,\delta_{\mathbf{xx'},i} - \mathscr{H}_i^{\mathrm{m}}(x')\,\delta_{\mathbf{xx'},j} + \partial_n \partial'_q \left[h_{injq}(x)\,\delta_{\mathbf{xx'}}\right]$$

$$h_{injq}(x) = \left[-\hat{S}_{q)(n}\mathscr{P}_{i)(j} - \delta^{kl}\frac{p_k \hat{S}_{l(n}\mathscr{P}_{i)(j} p_{q)}}{(np)(m-np)} + \delta^{kl}\frac{p_k \hat{S}_{l(q}\mathscr{P}_{j)(i} p_{n)}}{(np)(m-np)} \right] \delta$$

$$\mathscr{P}_{ij} \equiv \delta_{ij} - \frac{p_i p_j}{(np)^2}$$

$$\mathscr{H}^{\mathrm{m}}(x) = T^{00}\,, \quad \mathscr{H}_i^{\mathrm{m}}(x) = T^{0i}$$

- Local version of the Poincaré algerbra.
- Minkowski limit of the gravitational constraint algebra.
- Dirac field also has $h_{injq}(x) \neq 0$.

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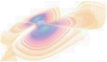
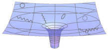




Order-by-Order Construction

- Hamiltonian $\hat{=}$ energy depending on canonical variables.
- Need to find canonical variables!
- Symmetries of the action:

$$P_i = \sum_a \hat{p}_{ai} + P_i^{\text{field}}\,, \qquad J_i = \sum_a \left[\varepsilon_{ijk} \hat{z}_a^j \hat{p}_{ak} + S_{a(i)} \right] + J_i^{\text{field}}$$

- Global Poincaré algebra:

$$\begin{aligned}
&\{P_i, P_j\} = 0\,, && \{P_i, H\} = 0\,, && \{J_i, H\} = 0 \\
&\{J_i, P_j\} = \varepsilon_{ijk} P_k\,, && \{J_i, J_j\} = \varepsilon_{ijk} J_k\,, && \{J_i, G_j\} = \varepsilon_{ijk} G_k \\
&\{G_i, P_j\} = H \delta_{ij}\,, && \{G_i, H\} = P_i\,, && \{G_i, G_j\} = -\varepsilon_{ijk} J_k
\end{aligned}$$

Canonical Variables at 2PN

- Calculate $\mathscr{H}_i^{\text{matter}}$:

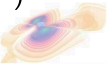
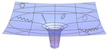
$$\mathscr{H}_i^{\text{matter}} = -\sqrt{\gamma} T_{i\nu} n^\nu$$

- Define canonical momentum $\hat{p}_i$ as:

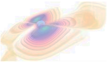
$$\hat{p}_i = \int d^3\mathbf{x}\, \mathscr{H}_i^{\text{matter}}$$

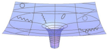
- Define spin $\hat{S}_{ij} = e_{i(k)} e_{j(l)} \varepsilon_{klm} \hat{S}_{(m)}$ such that $\hat{\mathbf{S}}^2 = \text{const.}$
- $\hat{\mathbf{z}}$ and $e_{i(k)}$ fixed by

$$J_{ij} = \hat{z}^i \hat{p}_j - \hat{z}^j \hat{p}_i + \varepsilon_{ijm} \hat{S}_{(m)} = \int d^3\mathbf{x} \left(x^i \mathscr{H}_j^{\text{matter}} - x^j \mathscr{H}_i^{\text{matter}} \right)$$

Outline





Leading-Order (LO)

- LO Spin-Orbit Hamiltonian:

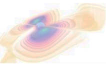
$$H_{SO}^{LO} = \sum_a \sum_{b \neq a} \frac{1}{\hat{r}_{ab}^2} (\hat{\mathbf{S}}_a \times \hat{\mathbf{n}}_{ab}) \cdot \left[\frac{3 m_b}{2 m_a} \hat{\mathbf{p}}_a - 2 \hat{\mathbf{p}}_b \right]$$

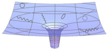
- LO Spin_1-Spin_2 Hamiltonian:

$$H_{S_1 S_2}^{LO} = \sum_a \sum_{b \neq a} \frac{1}{2 \hat{r}_{ab}^3} \left[3 (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab}) - (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{S}}_b) \right]$$

- Center of mass vector:

$$\mathbf{G}_{SO}^{LO} = \sum_a \frac{1}{2 m_a} (\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \,, \qquad \mathbf{G}_{S_1 S_2}^{LO} = 0$$





NLO Spin-Orbit

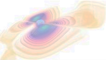
First derived: Damour, Jaranowski, and Schäfer (2008)

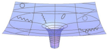
$$
\begin{aligned}
H_{\mathrm{SO}}^{\mathrm{NLO}} = & -\frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2}\left[\frac{5m_2\hat{\mathbf{p}}_1^2}{8m_1^3} + \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)}{4m_1^2} - \frac{3\hat{\mathbf{p}}_2^2}{4m_1 m_2}\right. \\
& \left. + \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{4m_1^2} + \frac{3(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})^2}{2m_1 m_2}\right] \\
& + \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2}\left[\frac{(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)}{m_1 m_2} + \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{m_1 m_2}\right] \\
& + \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{p}}_2)}{\hat{r}_{12}^2}\left[\frac{2(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{m_1 m_2} - \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})}{4m_1^2}\right] \\
& - \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3}\left[\frac{11m_2}{2} + \frac{5m_2^2}{m_1}\right] \\
& + \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3}\left[6m_1 + \frac{15m_2}{2}\right] + (1 \leftrightarrow 2)
\end{aligned}
$$

NLO Spin$_1$-Spin$_2$

Partial result: Porto and Rothstein (2006)

$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_1S_2} = {} & \frac{1}{2m_1m_2\hat{r}_{12}^3}\Big[\tfrac{3}{2}((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12})+\tfrac{1}{2}(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2) \\
& +6((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12})-\tfrac{1}{2}(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_1) \\
& -15(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})+(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_2) \\
& -3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)+3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12}) \\
& +3(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})+3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12}) \\
& +3(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})-3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})\Big] \\
& +\frac{3}{2m_1^2\hat{r}_{12}^3}\Big[-((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12}) \\
& +(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})^2-(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})\Big] \\
& +\frac{3}{2m_2^2\hat{r}_{12}^3}\Big[-((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12}) \\
& +(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})^2-(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})\Big] \\
& +\frac{6(m_1+m_2)}{\hat{r}_{12}^4}\Big[(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)-2(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})\Big]
\end{aligned}
$$



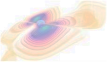


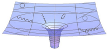
NLO Center of Mass

$$\mathbf{G}^{\text{NLO}}_{\text{SO}} = -\sum_a \frac{\hat{\mathbf{p}}_a^2}{8m_a^3}(\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a)$$
$$+\sum_a \sum_{b\neq a} \frac{m_b}{4m_a \hat{r}_{ab}} \left[((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a)\cdot \hat{\mathbf{n}}_{ab}) \frac{5\hat{\mathbf{z}}_a + \hat{\mathbf{z}}_b}{\hat{r}_{ab}} - 5(\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \right]$$
$$+\sum_a \sum_{b\neq a} \frac{1}{\hat{r}_{ab}} \left[\frac{3}{2}(\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a) - \frac{1}{2}(\hat{\mathbf{n}}_{ab} \times \hat{\mathbf{S}}_a)(\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab}) \right.$$
$$\left. - ((\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a)\cdot \hat{\mathbf{n}}_{ab}) \frac{\hat{\mathbf{z}}_a + \hat{\mathbf{z}}_b}{\hat{r}_{ab}} \right]$$

$$\mathbf{G}^{\text{NLO}}_{\text{SS}} = \frac{1}{2}\sum_a \sum_{b\neq a} \left\{ \left[3(\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab}) - (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{S}}_b) \right] \frac{\hat{\mathbf{z}}_a}{\hat{r}_{ab}^3} + (\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab}) \frac{\hat{\mathbf{S}}_a}{\hat{r}_{ab}^2} \right\}$$

$\Rightarrow$ Poincaré algebra is fulfilled.





Outline

since 1558

The Stress-Energy Tensor with Quadrupole

- Ansatz:

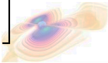
$$\sqrt{-g}T^{\mu\nu} = \int d\tau \left[t^{\mu\nu}\delta_{(4)} + (t^{\mu\nu\alpha}\delta_{(4)})_{||\alpha} + (t^{\mu\nu\alpha\beta}\delta_{(4)})_{||\alpha\beta} \right]$$

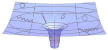
- Evaluate $T^{\mu\nu}{}_{||\nu} = 0$:

$$\frac{D(S^{\mu\nu})}{D\tau} = 2p^{[\mu}u^{\nu]} + \frac{4}{3}\mathrm{R}^{[\mu}{}_{\rho\beta\alpha}J^{\nu]\rho\beta\alpha}$$

$$\frac{Dp_\mu}{D\tau} = -\frac{1}{2}\mathrm{R}_{\mu\rho\beta\alpha}u^\rho S^{\beta\alpha} - \frac{1}{6}\mathrm{R}_{\nu\rho\beta\alpha;\mu}J^{\nu\rho\beta\alpha}$$

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \Big[u^{(\mu}p^{\nu)}\delta_{(4)} + \frac{1}{3}\mathrm{R}^{(\mu}{}_{\rho\beta\alpha}J^{\nu)\rho\beta\alpha}\delta_{(4)} + (u^{(\mu}S^{\nu)\alpha}\delta_{(4)})_{||\alpha} - \frac{2}{3}(J^{\mu\alpha\beta\nu}\delta_{(4)})_{||(\alpha\beta)} \Big]$$





- Mass quadrupole $I^{\mu\nu}$: $J^{\nu\rho\beta\alpha} = -3u^{[\nu}I^{\rho][\beta}u^{\alpha]}$
- Spin-squared ansatz for $I^{\mu\nu}$.

Full Hamiltonian relevant for Poincaré algebra up to 2PN:

$$H = H_N + H_{1PN} + H_{2PN} + H_{SO}^{1PN} + H_{SO}^{2PN} + H_{S^2} + H_{S^3 p} + H_{S^2 p^2} + H_{S^4}$$

- Source terms in canonical variables sufficient for $H_{S_2^2 S_1 p_1}$, $H_{S_2^3 p_1}$, $H_{S_1^3 p_2}$, $H_{S_1^2 S_2 p_2}$, $H_{S_1^2 S_2^2}$, $H_{S_1 S_2^3}$, and $H_{S_2 S_1^3}$ were obtained from the Kerr-metric in ADM coordinates.
- Ansatzes for $H_{S_1^2 p^2}$, $H_{S_2^2 p^2}$, $H_{S_1^3 p_1}$, $H_{S_2^3 p_2}$, $H_{S_1^2 S_2 p_1}$, $H_{S_2^2 S_1 p_2}$, $H_{S_1^4}$, and $H_{S_2^4}$ are fixed up to canonical transformation by $\{G_i, H\} = P_i$.
- The $\hat{p}_i$-independent part of the Hamilton constraint is needed to fix these remaining degrees of freedom and to get $H_{G^2 S^2}$.

Ansatz for $\hat{p}_i = 0$ Source Terms

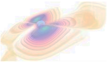
$$\mathscr{H}^{\text{matter}}_{S_1^2,\,\hat{p}_i=0} = \frac{c_1}{m_1}\left(Q_1^{ij}\hat{\delta}_1\right)_{;ij} + \frac{c_2}{m_1}\mathrm{R}_{ij}Q_1^{ij}\hat{\delta}_1 + \frac{c_3}{m_1}\hat{\mathbf{S}}_1^2\left(\gamma^{ij}\delta_1\right)_{;ij} + \frac{c_4}{m_1}\mathrm{R}\hat{\mathbf{S}}_1^2\hat{\delta}_1$$
$$+ \frac{1}{8m_1}g_{mn}\gamma^{pj}\gamma^{ql}\gamma^{mi}{}_{,p}\gamma^{nk}{}_{,q}\hat{S}_{1ij}\hat{S}_{1kl}\hat{\delta}_1 + \frac{1}{4m_1}\left(\gamma^{ij}\gamma^{mn}\gamma^{kl}{}_{,m}\hat{S}_{1ln}\hat{S}_{1jk}\hat{\delta}_1\right)_{,i}$$

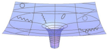
$$Q_1^{ij} = \gamma^{ik}\gamma^{jl}\gamma^{mn}\hat{S}_{1km}\hat{S}_{1nl} - \frac{2}{3}\gamma^{ij}\hat{\mathbf{S}}_1^2$$

- 3-dim. covariant, as $\hat{p}_i$ is not:

$$\hat{p}_i = p_i - \frac{1}{2}g_{ij}\gamma^{lm}\gamma^{kj}{}_{,m}\hat{S}_{kl} + \dots$$

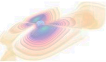
- Terms like $Q^{ij}_{1;k}\hat{\delta}_1$ or $Q_1^{ij}\hat{\delta}_{1;k}$ can not appear.
- Kerr metric $\Rightarrow$ $c_1 = -\frac{1}{2}$ and $c_2 = 0$.
- c_3 and c_4 do not contribute to the Hamiltonian.

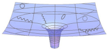




NLO Spin$_1$-Spin$_1$ for Black Holes

$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_1^2} = \frac{1}{\hat{r}_{12}^3}\Bigg[& \frac{m_2}{4m_1^3}\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{S}}_1\right)^2 + \frac{3m_2}{8m_1^3}\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12}\right)^2\hat{\mathbf{S}}_1^2 - \frac{3m_2}{8m_1^3}\hat{\mathbf{p}}_1^2\left(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12}\right)^2 \\
& - \frac{3m_2}{4m_1^3}\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12}\right)\left(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12}\right)\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{S}}_1\right) - \frac{3}{4m_1 m_2}\hat{\mathbf{p}}_2^2\hat{\mathbf{S}}_1^2 \\
& + \frac{9}{4m_1 m_2}\hat{\mathbf{p}}_2^2\left(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12}\right)^2 + \frac{3}{4m_1^2}\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2\right)\hat{\mathbf{S}}_1^2 \\
& - \frac{9}{4m_1^2}\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2\right)\left(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12}\right)^2 + \frac{3}{4m_1^2}\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12}\right)\left(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12}\right)\hat{\mathbf{S}}_1^2 \\
& - \frac{3}{2m_1^2}\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12}\right)\left(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{S}}_1\right)\left(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12}\right) \\
& + \frac{3}{m_1^2}\left(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12}\right)\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{S}}_1\right)\left(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12}\right) \\
& - \frac{15}{4m_1^2}\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12}\right)\left(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12}\right)\left(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12}\right)^2\Bigg] \\
- \frac{m_2}{\hat{r}_{12}^4}\Bigg[& \frac{9}{2}\left(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12}\right)^2 - \frac{5}{2}\hat{\mathbf{S}}_1^2 + \frac{7m_2}{m_1}\left(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12}\right)^2 - \frac{3m_2}{m_1}\hat{\mathbf{S}}_1^2\Bigg]
\end{aligned}
$$





Thank you for your attention!