

Canonical formulation of spin in general relativity and application to post-Newtonian approximations

Jan Steinhoff Steven Hergt Gerhard Schäfer



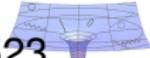
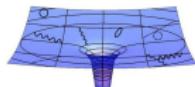
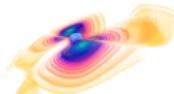
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DFG : SFB/TR7 “Gravitational Wave Astronomy” and GRK 1523

Outline

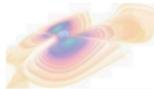
1 Introduction

2 Action Approach

3 Order-by-Order Construction

4 Results Linear in Spin

5 Higher Orders in Spin



Canonical Formulations

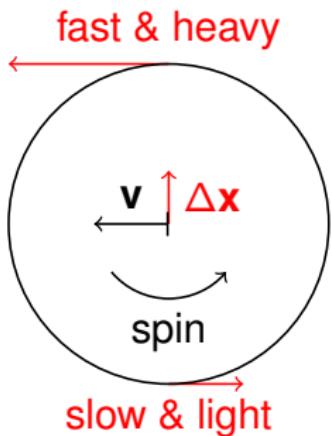
- Why?
 - Compact objects with spin interesting for GW astronomy.
 - Canonical formalism efficient for calculating conservative dynamics.
 - Consistency check ...
- Spinning test-bodies coupled to gravity:
 - H. P. Künzle, JMP **13**, 739 (1972).
 - K. Yee and M. Bander, PRD **48**, 2797 (1993).
 - E. Barausse, E. Racine, and A. Buonanno, PRD **80**, 104025 (2009).
- Dirac field coupled to gravity:
 - P. A. M. Dirac, in *Recent Developments in GR* (1962).
 - T. W. B. Kibble, JMP **4**, 1433 (1963).
 - S. Deser and C. J. Isham, PRD **14**, 2505 (1976).
 - J. Gegeniau and M. Henneaux, GRG **8**, 611 (1977).
 - J. E. Nelson and C. Teitelboim, Ann. Phys. (N.Y.) **116**, 86 (1978).



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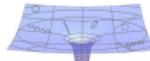
Spin in Special Relativity



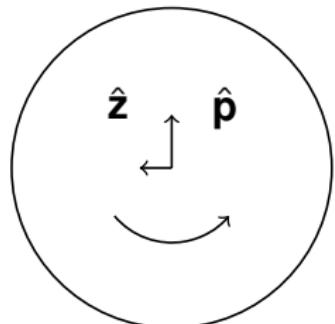
- Spin is a 4-tensor $S^{\mu\nu}$:
 - Spin is $S^{ij} = \epsilon^{ijk} S_k$.
 - Mass dipole is S^{i0} .
- Different mass centers.
- Need spin supplementary condition (SSC):
 - Møller SSC: $\tilde{S}^{\mu 0} = 0$
 - Covariant SSC: $S^{\mu\nu} p_\nu = 0$
 - **Newton-Wigner (canonical) SSC:**
 $m\hat{S}^{\mu 0} + \hat{S}^{\mu\nu} p_\nu = 0$
- In covariant SSC, with position \mathbf{z} :
- In Newton-Wigner SSC:

$$\{z^i, z^j\} = \frac{S^{ij}}{m^2} - \frac{p^i S^{0j} - p^j S^{0i}}{m^2 p^0}, \quad \dots$$

$$\{\hat{z}^i, p_j\} = \delta_{ij}, \quad \{\hat{S}_i, \hat{S}_j\} = \epsilon_{ijk} \hat{S}_k$$



Spin in Special Relativity



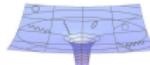
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- In Newton-Wigner SSC:

$$\{\hat{z}^i, p_j\} = \delta_{ij}, \quad \{\hat{S}_i, \hat{S}_j\} = \epsilon_{ijk} \hat{S}_k$$



(3+1)-Decomposition

- Normal vector:

$$n_\mu = (-N, 0, 0, 0), \quad n^\mu = \frac{1}{N}(1, -N^i), \quad n_\mu n^\mu = -1$$

- Projector:

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}, \quad g_{ij} = \gamma_{ij}, \quad \gamma_{ik}\gamma^{kj} = \delta_{ij}$$

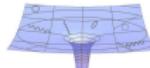
- Extrinsic curvature:

$$K_{ij} \equiv -n_{(i||j)}$$

$$\pi^{ij} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{lj} - \gamma^{ij}\gamma^{kl})K_{kl}$$



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Point-Mass Action in ADM Form

$$W = \int dt \cancel{p_i \dot{z}^i} + \int d^4x \left[\frac{1}{16\pi} \cancel{\pi^{ij} \gamma_{ij,0}} - H \right]$$
$$H = \int d^3x (N \mathcal{H} - N^i \mathcal{H}_i) + \underbrace{\frac{1}{16\pi} \oint d^2s_i (\gamma_{ij,j} - \gamma_{jj,i})}_{\text{ADM energy } E[\gamma_{ij}]}$$

- Constraint equations:

$$0 = \mathcal{H} \equiv -\frac{1}{16\pi\sqrt{\gamma}} \left[\gamma R + \frac{1}{2} \left(\gamma_{ij} \pi^{ij} \right)^2 - \gamma_{ij} \gamma_{kl} \pi^{ik} \pi^{jl} \right] + \mathcal{H}^{\text{matter}}$$

$$0 = \mathcal{H}_i \equiv \frac{1}{8\pi} \gamma_{ij} \pi^{jk}{}_{;k} + \mathcal{H}_i^{\text{matter}}$$

- Source terms are related to the stress-energy tensor $T^{\mu\nu}$:

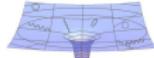
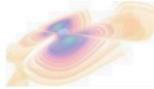
$$\mathcal{H}^{\text{matter}} \equiv \sqrt{\gamma} T_{\mu\nu} n^\mu n^\nu = \sqrt{m^2 + \gamma^{ij} p_i p_j} \delta$$

$$\mathcal{H}_i^{\text{matter}} \equiv -\sqrt{\gamma} T_{i\nu} n^\nu = p_i \delta$$

$$\delta \equiv \delta(x^i - z^i)$$



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ADM Canonical Formalism

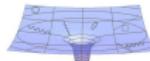
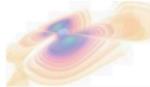
- Hamiltonian without gauge fixing:

$$H[z^i, p_i, \gamma_{ij}, \pi^{ij}] = \int d^3\mathbf{x} (N\mathcal{H} - N^i \mathcal{H}_i) + E[\gamma_{ij}]$$
$$E[\gamma_{ij}] = \frac{1}{16\pi} \oint d^2 s_i (\gamma_{ij,j} - \gamma_{jj,i})$$

- ADM Hamiltonian (Hamiltonian in ADMTT gauge)
 $\hat{=}$ ADM energy depending on canonical variables:

$$H_{\text{ADM}} = E[z^i, p_i, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}] = -\frac{1}{16\pi} \int d^3\mathbf{x} \Delta \phi$$
$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\text{TT}}, \quad \pi^{ii} = 0$$

- Matter only Hamiltonian: Elimination of h_{ij}^{TT} and π_{TT}^{ij} .



Outline

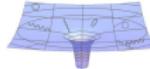
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Non-Relativistic Spherical Top

without center-of-mass motion

- Body fixed coordinates $z^{[i]}$: $z^i(t) = \Lambda_{[j]i}(t)z^{[j]}$, $\Lambda_{[k]i}\Lambda_{[k]j} = \delta_{ij}$
- Angular velocity tensor: $\Omega^{ij} = \epsilon_{ijk}\Omega^k = \Lambda_{[k]i}\dot{\Lambda}_{[k]j}$
- Spin tensor: $S_{ij} = 2\frac{\partial L}{\partial \Omega^{ij}}$
- Legendre transformed:

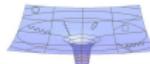
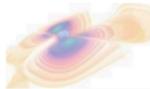
$$L = \frac{1}{2} \textcolor{red}{S_{ij}\Omega^{ij}} - H[\Lambda_{[i]j}, S_{ij}]$$

- Trick: Use $\delta\theta_{ij} = -\delta\theta_{ji} = \Lambda_{[k]i}\delta\Lambda_{[k]j}$ as independent variations.
- Usual Poisson brackets for the spin:

$$\{\hat{S}_i, \hat{S}_j\} = \epsilon_{ijk} \hat{S}_k$$



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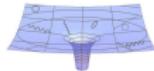
Relativistic Spherical Top

E.g., Hanson and Regge (1974)

- No rigid bodies.
- Mathematical abstraction: Top is
 - Worldline with Lorentz-matrix $\Lambda_{A\mu}$: $\eta^{AB}\Lambda_{A\mu}\Lambda_{B\nu} = \eta_{\mu\nu}$
 - $\Lambda_{A\mu}$ is pure rotation in rest-frame: $\Lambda^{[i]\mu} p_\mu = 0$
- Angular velocity tensor: $\Omega^{\mu\nu} = \Lambda_A{}^\mu \frac{d\Lambda^{Av}}{d\tau}$
- Spin tensor: $S_{\mu\nu} = 2 \frac{\partial L}{\partial \Omega^{\mu\nu}}$
- Associated SSC: $S_{\mu\nu} p^\mu = 0$



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Minimal Coupling

- Matter Lagrangian density and constraints:

$$\mathcal{L}_M = \int d\tau \left[p_\mu u^\mu + \frac{1}{2} S_{ab} \Omega^{ab} \right] \delta_{(4)}$$

$$\Omega^{ab} = \Lambda_A{}^a \frac{D\Lambda^{Ab}}{d\tau} = \Lambda_A{}^a \left[\frac{d\Lambda^{Ab}}{d\tau} - \Lambda^A{}_c \omega_\mu{}^{cb} u^\mu \right]$$

$$S_{ab} p^b = 0, \quad \Lambda^{[i]}{}^a p_a = 0, \quad p_\mu p^\mu + m^2 = 0$$

- Problem with metric variation:

$$\Lambda_{A\mu} \Lambda^A{}_v = g_{\mu v} \quad \leftrightarrow \quad \gamma_\mu \gamma_v + \gamma_v \gamma_\mu = 2g_{\mu v}$$

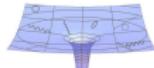
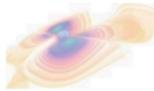
- Variate Λ^{Aa} and tetrad $e_{a\mu}$, $\Lambda^A{}_\mu = \Lambda^{Aa} e_{a\mu}$:

$$\Lambda_{Aa} \Lambda^A{}_b = \eta_{ab} \quad \leftrightarrow \quad \gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}$$

$$\delta \theta_{ab} = -\delta \theta_{ab} = \Lambda^C{}_a \delta \Lambda_{Cb}$$



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Result of Variation

- Approximated linear in spin.
- Field equations with stress-energy tensor (Mathisson, Tulczyjew):

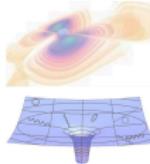
$$\sqrt{-g} T^{\mu\nu} = \int d\tau \left[mu^\mu u^\nu \delta_{(4)} - (S^{\alpha(\mu} u^{\nu)} \delta_{(4)})_{||\alpha} \right]$$
$$\delta_{(4)} \equiv \delta(x^\mu - z^\mu)$$

- EOM (Mathisson, Papapetrou):

$$\frac{DS^{\mu\nu}}{d\tau} = 0, \quad \frac{Dp_\mu}{d\tau} = \frac{1}{2} S^{\lambda\nu} u^\gamma R_{\mu\gamma\nu\lambda}$$

- Original derivation: Evaluate $T^{\mu\nu}_{||\nu} = 0$ with ansatz

$$\sqrt{-g} T^{\mu\nu} = \int d\tau \left[t^{\mu\nu} \delta_{(4)} + (t^{\mu\nu\alpha} \delta_{(4)})_{||\alpha} \right]$$



Reduction of the Matter Part

- Solve matter constraints, Schwinger time gauge $e_{(0)\mu} = n_\mu$, $\tau = t$.
- Variable redefinitions:

$$z^i = \hat{z}^i - \frac{nS^i}{m-np}, \quad np = -\sqrt{m^2 + \gamma^{ij} p_i p_j}$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i n S_j}{m-np} + \frac{p_j n S_i}{m-np}, \quad n S_i = -\frac{p_k \gamma^{kj} \hat{S}_{ji}}{m}$$

$$\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)} \left(\delta_{kj} + \frac{p_{(k)} p_{(j)}}{m(m-np)} \right), \quad \gamma_{ik} \gamma_{jl} A^{kl} = \frac{1}{2} \hat{S}_{ij} + \frac{mp_{(i)} n S_{j)}}{np(m-np)}$$

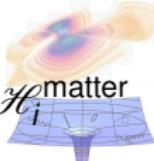
$$\hat{p}_i = p_i + K_{ij} n S^j + A^{kl} e_{(j)k} e_{l,i}^{(j)} - \left(\frac{1}{2} S_{kj} + \frac{p_{(k)} n S_{j)}}{np} \right) \Gamma^{kj}_i$$

- Matter Lagrangian density now, with $\hat{\Omega}^{(i)(j)} = \hat{\Lambda}_{[k]}^{(i)} \dot{\hat{\Lambda}}^{[k](j)}$,

$$\mathcal{L}_M = A^{ij} e_{(k)i} e_{j,0}^{(k)} \hat{\delta} + \hat{p}_i \dot{\hat{z}}^i \hat{\delta} + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} \hat{\delta} - N \mathcal{H}^{\text{matter}} + N^i \mathcal{H}_i^{\text{matter}}$$



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ADM Formalism with Spin

- Legendre transformation for gravitational field.
- Spatial symmetric gauge (Kibble 1963): $e_{(i)j} = e_{ij} = e_{ji}$

$$e_{ij} e_{jk} = \gamma_{ik} \quad \Rightarrow \quad (e_{ij}) = \sqrt{(\gamma_{ij})}$$

- Action:

$$W = \frac{1}{16\pi} \int d^4x \hat{\pi}^{ij} \gamma_{ij,0} + \int dt \left[\hat{p}_i \dot{z}^i + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} - H \right]$$

$$H = \int d^3x (N \mathcal{H} - N^i \mathcal{H}_i) + E[\gamma_{ij}]$$

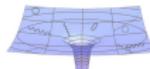
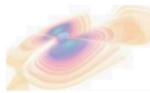
- Canonical field momentum:

$$\hat{\pi}^{ij} = \pi^{ij} + 8\pi A^{(ij)} \hat{\delta} + 16\pi B_{kl}^{ij} A^{[kl]} \hat{\delta}$$

$$2B_{kl}^{ij} = e_{mk} \frac{\partial e_{ml}}{\partial \gamma_{ij}} - e_{ml} \frac{\partial e_{mk}}{\partial \gamma_{ij}}$$



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Full Reduction

- Solve field constraints in ADMTT gauge.
- Fully reduced action:

$$W = \frac{1}{16\pi} \int d^4x \hat{\pi}_{\text{TT}}^{ij} h_{ij,0}^{\text{TT}} + \int dt \left[\hat{p}_i \dot{\hat{z}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} - H_{\text{ADM}} \right]$$
$$H_{\text{ADM}} = E[\hat{z}^i, \hat{p}_i, \hat{S}_{(i)(j)}, h_{ij}^{\text{TT}}, \hat{\pi}^{ij\text{TT}}]$$

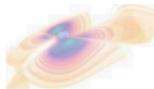
- Fundamental equal-time Poisson brackets:

$$\{\hat{z}^i, \hat{p}_j\} = \delta_{ij}, \quad \{\hat{S}_{(i)}, \hat{S}_{(j)}\} = \epsilon_{ijk} \hat{S}_{(k)}$$
$$\{h_{ij}^{\text{TT}}(\mathbf{x}, t), \hat{\pi}_{\text{TT}}^{kl}(\mathbf{x}', t)\} = 16\pi \delta_{ij}^{\text{TT}kl} \delta(\mathbf{x} - \mathbf{x}')$$

- Valid to all orders linear in spin.



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Conserved Quantities

- Energy: $E = H_{\text{ADM}}$
- Total linear and angular momentum:

$$P_i = \sum_a \hat{p}_{ai} - \frac{1}{16\pi} \int d^3x \hat{\pi}_{\text{TT}}^{kl} h_{kl,i}^{\text{TT}}$$
$$J_{ij} = \sum_a (\hat{z}_a^i \hat{p}_{aj} - \hat{z}_a^j \hat{p}_{ai}) + \sum_a \hat{S}_{a(i)(j)}$$
$$- \frac{1}{16\pi} \int d^3x (x^i \hat{\pi}_{\text{TT}}^{kl} h_{kl,j}^{\text{TT}} - x^j \hat{\pi}_{\text{TT}}^{kl} h_{kl,i}^{\text{TT}})$$
$$- \frac{1}{16\pi} \int d^3x 2(\hat{\pi}_{\text{TT}}^{ik} h_{kj}^{\text{TT}} - \hat{\pi}_{\text{TT}}^{jk} h_{ki}^{\text{TT}})$$

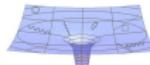
- Boost: $J^{i0} \equiv K^i \equiv G^i - t P^i$

With center-of-mass vector:

$$G^i = -\frac{1}{16\pi} \int d^3x x^i \Delta\phi$$



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Stress-Energy-Tensor Algebra (Minkowski)

$$\{\mathcal{H}^m(x), \mathcal{H}^m(x')\} = -\mathcal{H}_i^m(x)\delta_{xx',i} - \mathcal{H}_i^m(x')\delta_{xx',i}$$

$$\{\mathcal{H}_i^m(x), \mathcal{H}^m(x')\} = -\mathcal{H}^m(x)\delta_{xx',i} - T_{ij}(x')\delta_{xx',j}$$

$$\{\mathcal{H}_i^m(x), \mathcal{H}_j^m(x')\} = -\mathcal{H}_j^m(x)\delta_{xx',i} - \mathcal{H}_i^m(x')\delta_{xx',j} + \partial_n \partial'_q [h_{injq}(x)\delta_{xx'}]$$

$$h_{injq}(x) = \left[-\hat{S}_{q)(n}\mathcal{P}_i)(j} - \delta^{kl} \frac{\hat{p}_k \hat{S}_{l(n}\mathcal{P}_i)(j}\hat{p}_q)}{n\hat{p}(m-n\hat{p})} + \delta^{kl} \frac{\hat{p}_k \hat{S}_{l(q}\mathcal{P}_j)(i}\hat{p}_{n)}}{n\hat{p}(m-n\hat{p})} \right] \hat{\delta}$$

$$\mathcal{P}_{ij} \equiv \delta_{ij} - \frac{\hat{p}_i \hat{p}_j}{n\hat{p}^2}$$

$$\mathcal{H}^m(x) = T^{00}, \quad \mathcal{H}_i^m(x) = T^{0i}$$

- Local version of the Poincaré algebra.
- Minkowski limit of the gravitational constraint algebra.
- Dirac field also has $h_{injq}(x) \neq 0$.



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1

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4

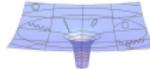
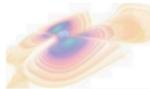
Results Linear in Spin

5

Higher Orders in Spin



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Order-by-Order Construction

- Hamiltonian $\hat{=}$ energy depending on canonical variables.
- Need to find canonical variables!
- Relate symmetry generators

$$P_i = \sum_a \hat{p}_{ai} + P_i^{\text{field}}, \quad J_i = \sum_a \left[\varepsilon_{ijk} \hat{z}_a^j \hat{p}_{ak} + \hat{S}_{a(i)} \right] + J_i^{\text{field}}$$

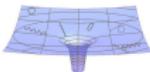
with surface integrals at spatial infinity.

- Global Poincaré algebra:

$$\{P_i, P_j\} = 0, \quad \{P_i, H\} = 0, \quad \{J_i, H\} = 0$$

$$\{J_i, P_j\} = \varepsilon_{ijk} P_k, \quad \{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, G_j\} = \varepsilon_{ijk} G_k$$

$$\{G_i, P_j\} = H \delta_{ij}, \quad \{G_i, H\} = P_i, \quad \{G_i, G_j\} = -\varepsilon_{ijk} J_k$$



Symmetry Generators

- ADM gauge conditions invariant under

$$x^i \rightarrow x^i + a^i + \omega^{ij} x^j, \quad a^i = \text{const}, \quad \omega^{ij} = \omega^{[ij]} = \text{const}$$

- Matter variables transform as

$$\hat{z}_a^i \rightarrow \hat{z}_a^i + a^i + \omega^{ij} \hat{z}_a^j$$

$$\hat{p}_{ai} \rightarrow \hat{p}_{ai} + \omega^{ij} \hat{p}_{aj}$$

$$\hat{S}_{a(i)(j)} \rightarrow \hat{S}_{a(i)(j)} + \omega^{im} \hat{S}_{a(m)(j)} + \omega^{jm} \hat{S}_{a(i)(m)}$$

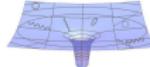
- Construct generators P_i and J_{ij} such that

$$A \rightarrow A + \frac{1}{2} \omega^{ij} \{A, J_{ji}\} + a^i \{A, P_i\}$$

$$\Rightarrow \quad \delta_{ij} = \{\hat{z}_a^i, P_j\} = \frac{\partial P_j}{\partial \hat{p}_{ai}} \quad \Rightarrow \quad P_i = \sum_a \hat{p}_{ai} + P_i^{\text{field}}$$

- Similarly

$$J_{ij} = \sum_a \left[\hat{z}_a^i \hat{p}_{aj} - \hat{z}_a^j \hat{p}_{ai} + \hat{S}_{a(i)(j)} \right] + J_{ij}^{\text{field}}$$



Canonical Variables at 2PN

- Calculate $\mathcal{H}_i^{\text{matter}}$:

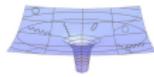
$$\mathcal{H}_i^{\text{matter}} = -\sqrt{\gamma} T_{i\nu} n^\nu$$

- Define canonical momentum \hat{p}_i as:

$$\hat{p}_i = \int d^3\mathbf{x} \mathcal{H}_i^{\text{matter}}$$

- Define spin $\hat{S}_{ij} = e_{i(k)} e_{j(l)} \epsilon_{klm} \hat{S}_{(m)}$ such that $\hat{\mathbf{S}}^2 = \text{const.}$
- $\hat{\mathbf{z}}$ and $e_{i(k)}$ fixed by

$$J_{ij} = \hat{z}^i \hat{p}_j - \hat{z}^j \hat{p}_i + \epsilon_{ijm} \hat{S}_{(m)} = \int d^3\mathbf{x} (x^i \mathcal{H}_j^{\text{matter}} - x^j \mathcal{H}_i^{\text{matter}})$$



Outline

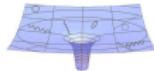
1 Introduction

2 Action Approach

3 Order-by-Order Construction

4 Results Linear in Spin

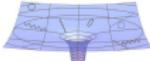
5 Higher Orders in Spin



Next-to-Leading Order (NLO) Spin-Orbit

See also: Tagoshi, Ohashi, Owen (2001); Faye, Blanchet, Buonanno (2006)
Hamiltonian first derived: Damour, Jaranowski, and Schäfer (2008)

$$\begin{aligned} H_{\text{SO}}^{\text{NLO}} = & - \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2} \left[\frac{5m_2 \hat{\mathbf{p}}_1^2}{8m_1^3} + \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)}{4m_1^2} - \frac{3\hat{\mathbf{p}}_2^2}{4m_1 m_2} \right. \\ & \quad \left. + \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})}{4m_1^2} + \frac{3(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})^2}{2m_1 m_2} \right] \\ & + \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2} \left[\frac{(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)}{m_1 m_2} + \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})}{m_1 m_2} \right] \\ & + \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{p}}_2)}{\hat{r}_{12}^2} \left[\frac{2(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})}{m_1 m_2} - \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})}{4m_1^2} \right] \\ & - \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3} \left[\frac{11m_2}{2} + \frac{5m_2^2}{m_1} \right] \\ & + \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3} \left[6m_1 + \frac{15m_2}{2} \right] + (1 \leftrightarrow 2) \end{aligned}$$



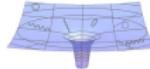
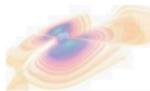
NLO Spin₁-Spin₂

Partial result: Porto and Rothstein (2006)

$$\begin{aligned} H_{S_1 S_2}^{\text{NLO}} = & \frac{1}{2m_1 m_2 \hat{r}_{12}^3} [\frac{3}{2} ((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12}) ((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) + \frac{1}{2} (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \\ & + 6((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12}) ((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) - \frac{1}{2} (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1) \\ & - 15(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_2) \\ & - 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) + 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) \\ & + 3(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) + 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \\ & + 3(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_2) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) - 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})] \\ & + \frac{3}{2m_1^2 \hat{r}_{12}^3} [- ((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12}) ((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) \\ & \quad + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})^2 - (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})] \\ & + \frac{3}{2m_2^2 \hat{r}_{12}^3} [- ((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) ((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12}) \\ & \quad + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})^2 - (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})] \\ & + \frac{6(m_1 + m_2)}{\hat{r}_{12}^4} [(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) - 2(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})] \end{aligned}$$



since 1388



NLO Center of Mass

$$\begin{aligned}\mathbf{G}_{\text{SO}}^{\text{NLO}} = & - \sum_a \frac{\hat{\mathbf{p}}_a^2}{8m_a^3} (\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \\ & + \sum_a \sum_{b \neq a} \frac{m_b}{4m_a \hat{r}_{ab}} \left[((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab}) \frac{5\hat{\mathbf{z}}_a + \hat{\mathbf{z}}_b}{\hat{r}_{ab}} - 5(\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \right] \\ & + \sum_a \sum_{b \neq a} \frac{1}{\hat{r}_{ab}} \left[\frac{3}{2} (\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a) - \frac{1}{2} (\hat{\mathbf{n}}_{ab} \times \hat{\mathbf{S}}_a) (\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab}) \right. \\ & \quad \left. - ((\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab}) \frac{\hat{\mathbf{z}}_a + \hat{\mathbf{z}}_b}{\hat{r}_{ab}} \right] \\ \mathbf{G}_{\text{S}_1 \text{S}_2}^{\text{NLO}} = & \frac{1}{2} \sum_a \sum_{b \neq a} \left\{ \left[3(\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab}) - (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{S}}_b) \right] \frac{\hat{\mathbf{z}}_a}{\hat{r}_{ab}^3} + (\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab}) \frac{\hat{\mathbf{S}}_a}{\hat{r}_{ab}^2} \right\}\end{aligned}$$



⇒ Poincaré algebra is fulfilled.

Outline

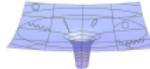
1 Introduction

2 Action Approach

3 Order-by-Order Construction

4 Results Linear in Spin

5 Higher Orders in Spin



The Stress-Energy Tensor with Quadrupole

- Ansatz:

$$\sqrt{-g} T^{\mu\nu} = \int d\tau \left[t^{\mu\nu} \delta_{(4)} + (t^{\mu\nu\alpha} \delta_{(4)})_{||\alpha} + (\textcolor{red}{t^{\mu\nu\alpha\beta}} \delta_{(4)})_{||(\alpha\beta)} \right]$$

- Evaluate $T^{\mu\nu}_{||\nu} = 0$: (with D. Puetzfeld)

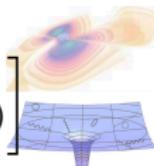
$$\frac{D(S^{\mu\nu})}{D\tau} = 2p^{[\mu} u^{\nu]} + \frac{4}{3} R_{\rho\beta\alpha}^{[\mu} \textcolor{red}{J^{\nu}\rho\beta\alpha}$$

$$\frac{Dp_\mu}{D\tau} = -\frac{1}{2} R_{\mu\rho\beta\alpha} u^\rho S^{\beta\alpha} - \frac{1}{6} R_{\nu\rho\beta\alpha;\mu} \textcolor{red}{J^{\nu\rho\beta\alpha}}$$

$$\begin{aligned} \sqrt{-g} T^{\mu\nu} = \int d\tau & \left[u^{(\mu} p^{\nu)} \delta_{(4)} + \frac{1}{3} R_{\rho\beta\alpha}^{(\mu} \textcolor{red}{J^{\nu)\rho\beta\alpha}} \delta_{(4)} \right. \\ & \left. + (u^{(\mu} S^{\nu)\alpha} \delta_{(4)})_{||\alpha} - \frac{2}{3} (\textcolor{red}{J^{\mu\alpha\beta\nu}} \delta_{(4)})_{||(\alpha\beta)} \right] \end{aligned}$$



since 1358



Quadrupole Deformation due to Spin

- Mass quadrupole $Q^{\mu\nu}$:

$$J^{\nu\rho\beta\alpha} = -3u^{[\nu}Q^{\rho]}{}^{\beta}u^{\alpha]}$$

- Spin-squared ansatz for $Q^{\mu\nu}$:

$$Q_{\mu\nu} = \frac{C_Q}{m_p} \left(S_{\mu\rho} S_{\nu}{}^{\rho} - \frac{1}{3} P_{\mu\nu} S^{\rho\sigma} S_{\rho\sigma} \right)$$

- For black holes $C_Q = 1$.
- Comparison with EFT approach (Porto, Rothstein, 2008):

$$L_{S^2} = \underbrace{-\frac{1}{2m} R_{\mu\nu\alpha\beta}^{(4)} S^{\rho\mu} S^{\alpha\beta} \frac{u^\nu u_\rho}{\sqrt{-u_\sigma u^\sigma}}}_{\text{SSC preserving}} - \underbrace{\frac{1}{2} R_{\alpha\mu\beta\nu}^{(4)} Q^{\alpha\beta} \frac{u^\mu u^\nu}{\sqrt{-u_\sigma u^\sigma}}}_{\text{quadrupole deformation}}$$

$$J^{\mu\nu\alpha\beta} = -6 \frac{\partial L_{S^2}}{\partial R_{\mu\nu\alpha\beta}^{(4)}} \quad \text{Bailey, Israel (1975)}$$

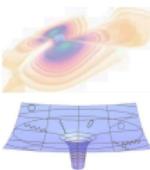
- K_{ij} in matter action problematic for canonical formulation.

NLO Spin₁-Spin₁ for Black Holes and Neutron Stars

$$\begin{aligned} H_{S_1^2}^{\text{NLO}} = & \frac{m_2}{m_1^3 \hat{r}_{12}^3} \left[\left(\frac{15}{4} - \frac{9}{2} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) + \left(\frac{5}{4} - \frac{5}{4} C_Q \right) \hat{\mathbf{p}}_1^2 \hat{\mathbf{S}}_1^2 \right. \\ & + \left(-\frac{9}{8} + \frac{3}{2} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \hat{\mathbf{S}}_1^2 + \left(-\frac{21}{8} + \frac{9}{4} C_Q \right) \hat{\mathbf{p}}_1^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\ & + \left. \left(-\frac{5}{4} + \frac{3}{2} C_Q \right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)^2 \right] + \frac{C_Q}{m_1 m_2 \hat{r}_{12}^3} \left[\frac{9}{4} \hat{\mathbf{p}}_2^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 - \frac{3}{4} \hat{\mathbf{p}}_2^2 \hat{\mathbf{S}}_1^2 \right] \\ & + \frac{1}{m_1^2 \hat{r}_{12}^3} \left[\left(-\frac{3}{2} + \frac{9}{2} C_Q \right) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) \right. \\ & + \left(-3 + \frac{3}{2} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) + \left(-\frac{3}{2} + \frac{9}{4} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \hat{\mathbf{S}}_1^2 \\ & + \left(\frac{3}{2} - \frac{3}{4} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \hat{\mathbf{S}}_1^2 + \left(3 - \frac{21}{4} C_Q \right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\ & - \left. \frac{15}{4} C_Q (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 + \left(\frac{3}{2} - \frac{3}{2} C_Q \right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) \right] \\ & + \frac{m_2}{\hat{r}_{12}^4} \left[\left(2 + \frac{1}{2} C_Q \right) \hat{\mathbf{S}}_1^2 - \left(3 + \frac{3}{2} C_Q \right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \right] \\ & + \frac{m_2^2}{m_1 \hat{r}_{12}^4} \left[(1 + 2 C_Q) \hat{\mathbf{S}}_1^2 - (1 + 6 C_Q) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \right] \end{aligned}$$



since 1558



Thank you for your attention

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