Canonical formulation of extended bodies in GR

and application to post-Newtonian approximations

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since 1558

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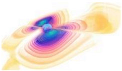

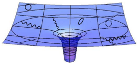

DFG: SFB/TR7 “Gravitational Wave Astronomy” and GRK 1523

Outline

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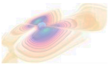
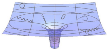
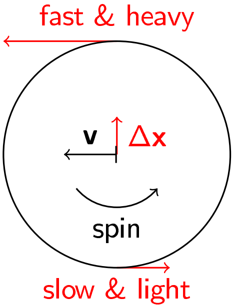






- Spinning object moving with velocity **v**.
- Shall have constant density in rest frame.
- Upper hemisphere faster than lower.
- Upper hemisphere more massive than lower.
- Center of mass displaced by $\Delta\mathbf{x}$.
- Spin depends on location of center.

- Description by means of a 4-tensor $S^{\mu\nu}$:
  - Spin is $S^{ij} = \varepsilon^{ijk} S_k$.
  - Mass dipole related to S^{0i}.
- Spin supplementary condition (SSC) fixates S^{0i} in terms of S^{ij}.

SSC in SR

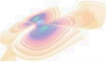
- Usefull SSCs are, with mass m and 4-momentum $p_\mu = m u_\mu$:
  - Møller SSC: $\tilde{S}^{0\mu} = 0$
  - Fokker-Synge-Pryce (covariant) SSC: $S^{\mu\nu} p_\nu = 0$
  - Newton-Wigner (canonical) SSC: $m\hat{S}^{0\mu} - \hat{S}^{\mu\nu} p_\nu = 0$
- Canonical structure depends on SSC, and can be complicated.
- In covariant SSC, with position **z**:

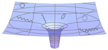
$$\{z^i(t), z^j(t)\} = \frac{S^{ij}}{m^2} - \frac{p^i S^{0j} - p^j S^{0i}}{m^2 p^0}\,, \quad \dots$$

- In Newton-Wigner SSC:

$$\{\hat{z}^i(t), p_j(t)\} = \delta_{ij}\,, \quad \{\hat{S}_i(t), \hat{S}_j(t)\} = \varepsilon_{ijk} \hat{S}_k(t)$$

$$\{\hat{\mathbf{S}}^2, \dots\} = 0 \quad \Rightarrow \quad \hat{\mathbf{S}}^2 = \text{const.}$$





Spin in GR

- We restrict to linear order in spin for now:
  - No deformation by spin or tidal forces included.
  - Linear order is universal.
- Stress-Energy Tensor in covariant SSC:

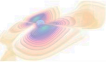
$$\sqrt{-g}\, T^{\mu\nu} = \int d\tau \left[m u^{\mu} u^{\nu} \delta_{(4)} - (S^{\alpha(\mu} u^{\nu)} \delta_{(4)})_{||\alpha} \right]$$

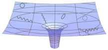
$$\delta_{(4)} \equiv \delta(x - z)\,, \quad \delta \equiv \delta(\mathbf{x} - \mathbf{z})$$

- EOM follow from $T^{\mu\nu}{}_{||\nu} = 0$:

$$\frac{DS^{\mu\nu}}{d\tau} = 0\,, \quad \frac{Dp_{\mu}}{d\tau} = \frac{1}{2} S^{\lambda\nu} u^{\gamma} R_{\mu\gamma\nu\lambda}$$

- Various actions are known (more later).





Perturbative Solution of Partial Differential Equations

Example:

- $\Delta f(\mathbf{x}) = a(\mathbf{x}) + b(\mathbf{x}) f(\mathbf{x}) + c(\mathbf{x})[f(\mathbf{x})]^2 + \dots$
- In general: vectors, other diff. op., derivatives on RHS, ...
- Perturbative expansion, e.g. $a = a_{(1)} + a_{(2)} + a_{(3)} + \dots$
- Leads to recursive equations:

$$\begin{aligned}
\Delta f_{(1)}(\mathbf{x}) &= a_{(1)}(\mathbf{x}) \\
\Delta f_{(2)}(\mathbf{x}) &= a_{(2)}(\mathbf{x}) + b_{(1)}(\mathbf{x}) f_{(1)}(\mathbf{x}) \\
\Delta f_{(3)}(\mathbf{x}) &= a_{(3)}(\mathbf{x}) + b_{(1)}(\mathbf{x}) f_{(2)}(\mathbf{x}) + b_{(2)}(\mathbf{x}) f_{(1)}(\mathbf{x}) \\
&\quad + c_{(1)}(\mathbf{x})[f_{(1)}(\mathbf{x})]^2 \\
&\vdots
\end{aligned}$$

- Delta sources $\Rightarrow$ Regularization, e.g., calculate in d dimensions.

(3+1)-Decomposition: Metric

- Decomposition of the metric:

$$g_{\mu\nu} = \begin{pmatrix} N^i N_i - N^2 & N_i \\ N_j & \gamma_{ij} \end{pmatrix}, \quad g^{00} = -\frac{1}{N^2}$$

- Normal vector of the 3-dim. hypersurfaces:

$$n_\mu = (-N, 0, 0, 0)$$

- Exterior curvature of the 3-dim. hypersurfaces:

$$K_{ij} \equiv -n_{(i||j)} = -N\Gamma^0_{ij}$$

$$\pi^{ij} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{lj} - \gamma^{ij}\gamma^{kl})K_{kl}$$

(3+1)-Decomposition: Field Equations

- Decomposition of the field equations:
  - Constraint equations:

$$0 = \mathscr{H} \equiv -\frac{1}{16\pi\sqrt{\gamma}}\left[\gamma \mathrm{R} + \frac{1}{2}\left(\gamma_{ij}\pi^{ij}\right)^2 - \gamma_{ij}\gamma_{kl}\pi^{ik}\pi^{jl}\right] + \mathscr{H}^{\text{matter}}$$

$$0 = \mathscr{H}_i \equiv \frac{1}{8\pi}\gamma_{ij}\pi^{jk}{}_{;k} + \mathscr{H}_i^{\text{matter}}$$

  - Evolution equations:

$$\gamma_{ij,0} = 2N\gamma^{-1/2}(\pi_{ij} - \tfrac{1}{2}\gamma_{ij}\gamma_{kl}\pi^{kl}) + N_{i;j} + N_{j;i}$$

$$\begin{aligned}\pi^{ij}{}_{,0} = {} & -N\sqrt{\gamma}(\mathrm{R}^{ij} - \tfrac{1}{2}\gamma^{ij}\mathrm{R}) + \tfrac{1}{2}N\gamma^{-1/2}\gamma^{ij}(\pi^{mn}\pi_{mn} - \tfrac{1}{2}(\gamma_{mn}\pi^{mn})^2) \\ & - 2N\gamma^{-1/2}(\gamma_{mn}\pi^{im}\pi^{nj} - \tfrac{1}{2}\gamma_{mn}\pi^{mn}\pi^{ij}) + \sqrt{\gamma}(N^{;ij} - \gamma^{ij}N^{;m}{}_{;m}) \\ & + (\pi^{ij}N^m)_{;m} - N^i{}_{;m}\pi^{mj} - N^j{}_{;m}\pi^{mi} + \tfrac{1}{2}N\gamma^{im}\gamma^{nj}\sqrt{\gamma}T_{mn}\end{aligned}$$

- Source terms are related to the stress-energy tensor $T^{\mu\nu}$ by:

$$\mathscr{H}^{\text{matter}} \equiv \sqrt{\gamma}T_{\mu\nu}n^\mu n^\nu$$

$$\mathscr{H}_i^{\text{matter}} \equiv -\sqrt{\gamma}T_{i\nu}n^\nu$$

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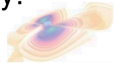
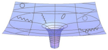




ADM Canonical Formalism

- Gauge independent Hamiltonian:

$$H[x_a^i, p_{ai}, \gamma_{ij}, \pi^{ij}] = \int d^3\mathbf{x}(N\mathscr{H} - N^i\mathscr{H}_i) + E[\gamma_{ij}]$$

$$E[\gamma_{ij}] = \frac{1}{16\pi}\oint d^2 s_i(\gamma_{ij,j} - \gamma_{jj,i})$$

- Hamiltonian in ADMTT gauge (ADM Hamiltonian)
  $\hat{=}$ ADM Energy depending on canonical variables:

$$H_{\mathrm{ADM}} = E[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}, \pi_{\mathrm{TT}}^{ij}] = -\frac{1}{16\pi}\int \mathrm{d}^3\mathbf{x}\,\Delta\phi$$

$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$

- Matter only Hamiltonian: Elimination of h_{ij}^{TT} and π_{TT}^{ij} by solving the evolution equations.

Global Poincaré Invariance: Generators

- Global Poincaré group is a consequence of asymptotic flatness.
- (3+1)-decomposition of P^{μ} and $J^{\mu\nu}$:
  - Energy: $E \equiv P^0$
  - Momentum: P^i
  - Angular momentum: $J^i \equiv \frac{1}{2}\varepsilon^{ijk} J_{jk}$
  - Boost: $J^{i0} \equiv K^i \equiv G^i - t\,P^i$
  - Center of mass: $X^i \equiv G^i / E$
- P_i, J_{ij}, E and G^i in ADMTT gauge (2PN):

$$P_i = \int d^3\mathbf{x}\,\mathscr{H}_i^{\text{matter}} \qquad J_{ij} = \int d^3\mathbf{x}\,(x^i\,\mathscr{H}_j^{\text{matter}} - x^j\,\mathscr{H}_i^{\text{matter}})$$

$$E = -\frac{1}{16\pi}\int \mathrm{d}^3\mathbf{x}\,\Delta\phi \qquad G^i = -\frac{1}{16\pi}\int \mathrm{d}^3\mathbf{x}\,x^i\,\Delta\phi$$

Global Poincaré Invariance: Algebra

- The algebra of global Poincaré invariance reads

$$
\begin{aligned}
&\{P_i, P_j\} = 0\,, && \{P_i, H\} = 0\,, && \{J_i, H\} = 0\,,\\
&\{J_i, P_j\} = \varepsilon_{ijk} P_k\,, && \{J_i, J_j\} = \varepsilon_{ijk} J_k\,, && \{J_i, G_j\} = \varepsilon_{ijk} G_k\,,\\
&\{G_i, P_j\} = H\delta_{ij}\,, && {\color{red}\{G_i, H\} = P_i}\,, && \{G_i, G_j\} = -\varepsilon_{ijk} J_k\,,
\end{aligned}
$$

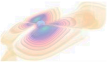
- Fundamental equal-time Poisson brackets:

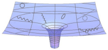
$$
\{\hat{z}_a^i, P_{aj}\} = \delta_{ij}\,, \quad \{\hat{S}_{a(i)}, \hat{S}_{a(j)}\} = \varepsilon_{ijk} \hat{S}_{a(k)}\,,
$$

$$
\{h_{ij}^{\mathrm{TT}}(\mathbf{x}, t), \pi^{kl\mathrm{TT}}(\mathbf{x}', t)\} = 16\pi \delta_{ij}^{\mathrm{TT}kl} \delta(\mathbf{x} - \mathbf{x}')\,,
$$

- It must hold:

$$
P_i = \sum_a P_{ai}\,, \quad J_i = \sum_a \left[\varepsilon_{ijk} \hat{z}_a^j P_{ak} + \hat{S}_{a(i)}\right] .
$$





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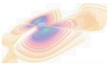
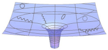




Identification of Canonical Variables

- Calculate $\mathscr{H}_i^{\text{matter}}$:

$$\mathscr{H}_i^{\text{matter}} = -\sqrt{\gamma} T_{i\nu} n^{\nu}$$

- Define canonical momentum P_i as:

$$P_i = \int d^3\mathbf{x}\, \mathscr{H}_i^{\text{matter}}$$

- Define spin $\hat{S}_{ij} = e_{i(k)} e_{j(l)} \varepsilon_{klm} \hat{S}_{(m)}$ such that $\mathbf{S}^2 = \text{const.}$ and

$$J_{ij} = \hat{z}^i P_j - \hat{z}^j P_i + \varepsilon_{ijm} \hat{S}_{(m)} = \int d^3\mathbf{x}\, (x^i \mathscr{H}_j^{\text{matter}} - x^j \mathscr{H}_i^{\text{matter}})$$

- Go over to canonical position variable **z** by a Lie shift (such that one has the Newton-Wigner SSC in flat space).

The Leading-Order (LO) in Spin

- LO Spin-Orbit Hamiltonian:

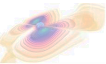
$$H_{\mathrm{SO}}^{\mathrm{LO}} = \sum_a \sum_{b \neq a} \frac{1}{r_{ab}^2} (\mathbf{S}_a \times \mathbf{n}_{ab}) \cdot \left[\frac{3 m_b}{2 m_a} \mathbf{P}_a - 2 \mathbf{P}_b \right]$$

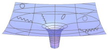
- LO Spin_1-Spin_2 Hamiltonian:

$$H_{\mathrm{SS}}^{\mathrm{LO}} = \sum_a \sum_{b \neq a} \frac{1}{2 r_{ab}^3} \left[3 (\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b) \right]$$

- Center of mass vector:

$$\mathbf{G}_{\mathrm{SO}}^{\mathrm{LO}} = \sum_a \frac{1}{2 m_a} (\mathbf{P}_a \times \mathbf{S}_a) \,, \qquad \mathbf{G}_{\mathrm{SS}}^{\mathrm{LO}} = 0$$

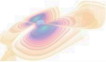


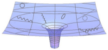


NLO Spin-Orbit Hamiltonian

First derived: Damour, Jaranowski, and Schäfer (2008)

$$
\begin{aligned}
H_{SO}^{NLO} = & -\frac{((\mathbf{P}_1 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^2}\left[\frac{5 m_2 \mathbf{P}_1^2}{8 m_1^3} + \frac{3(\mathbf{P}_1 \cdot \mathbf{P}_2)}{4 m_1^2} - \frac{3 \mathbf{P}_2^2}{4 m_1 m_2} \right. \\
& \left. + \frac{3(\mathbf{P}_1 \cdot \mathbf{n}_{12})(\mathbf{P}_2 \cdot \mathbf{n}_{12})}{4 m_1^2} + \frac{3(\mathbf{P}_2 \cdot \mathbf{n}_{12})^2}{2 m_1 m_2}\right] \\
& + \frac{((\mathbf{P}_2 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^2}\left[\frac{(\mathbf{P}_1 \cdot \mathbf{P}_2)}{m_1 m_2} + \frac{3(\mathbf{P}_1 \cdot \mathbf{n}_{12})(\mathbf{P}_2 \cdot \mathbf{n}_{12})}{m_1 m_2}\right] \\
& + \frac{((\mathbf{P}_1 \times \mathbf{S}_1)\cdot \mathbf{P}_2)}{r_{12}^2}\left[\frac{2(\mathbf{P}_2 \cdot \mathbf{n}_{12})}{m_1 m_2} - \frac{3(\mathbf{P}_1 \cdot \mathbf{n}_{12})}{4 m_1^2}\right] \\
& - \frac{((\mathbf{P}_1 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^3}\left[\frac{11 m_2}{2} + \frac{5 m_2^2}{m_1}\right] \\
& + \frac{((\mathbf{P}_2 \times \mathbf{S}_1)\cdot \mathbf{n}_{12})}{r_{12}^3}\left[6 m_1 + \frac{15 m_2}{2}\right] + (1 \leftrightarrow 2)
\end{aligned}
$$

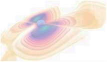


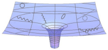


NLO Spin$_1$-Spin$_2$ Hamiltonian

Partial result: Porto and Rothstein (2006)

$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_1S_2} = \frac{1}{2m_1m_2r_{12}^3}\Big[&\tfrac{3}{2}((\mathbf{P}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{P}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) + \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{P}_1\cdot\mathbf{P}_2)\\
&+6((\mathbf{P}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{P}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) - \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{P}_2)(\mathbf{S}_2\cdot\mathbf{P}_1)\\
&-15(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{P}_1\cdot\mathbf{n}_{12})(\mathbf{P}_2\cdot\mathbf{n}_{12}) + (\mathbf{S}_1\cdot\mathbf{P}_1)(\mathbf{S}_2\cdot\mathbf{P}_2)\\
&-3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{P}_1\cdot\mathbf{P}_2) + 3(\mathbf{S}_1\cdot\mathbf{P}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{P}_1\cdot\mathbf{n}_{12})\\
&+3(\mathbf{S}_2\cdot\mathbf{P}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{P}_2\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_1\cdot\mathbf{P}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{P}_2\cdot\mathbf{n}_{12})\\
&+3(\mathbf{S}_2\cdot\mathbf{P}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{P}_1\cdot\mathbf{n}_{12}) - 3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{P}_1\cdot\mathbf{n}_{12})(\mathbf{P}_2\cdot\mathbf{n}_{12})\Big]\\
+\frac{3}{2m_1^2r_{12}^3}\Big[&-((\mathbf{P}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{P}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12})\\
&+(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{P}_1\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{P}_1)(\mathbf{P}_1\cdot\mathbf{n}_{12})\Big]\\
+\frac{3}{2m_2^2r_{12}^3}\Big[&-((\mathbf{P}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12})((\mathbf{P}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})\\
&+(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{P}_2\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{P}_2)(\mathbf{P}_2\cdot\mathbf{n}_{12})\Big]\\
+\frac{6(m_1+m_2)}{r_{12}^4}\Big[&(\mathbf{S}_1\cdot\mathbf{S}_2) - 2(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})\Big]
\end{aligned}
$$



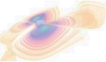


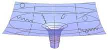
NLO Center of Mass

$$\mathbf{G}_{\mathrm{SO}}^{\mathrm{NLO}} = -\sum_a \frac{\mathbf{P}_a^2}{8m_a^3}(\mathbf{P}_a \times \mathbf{S}_a)$$
$$+\sum_a \sum_{b\neq a} \frac{m_b}{4m_a r_{ab}} \left[((\mathbf{P}_a \times \mathbf{S}_a)\cdot\mathbf{n}_{ab}) \frac{5\mathbf{z}_a + \mathbf{z}_b}{r_{ab}} - 5(\mathbf{P}_a \times \mathbf{S}_a) \right]$$
$$+\sum_a \sum_{b\neq a} \frac{1}{r_{ab}} \left[\frac{3}{2}(\mathbf{P}_b \times \mathbf{S}_a) - \frac{1}{2}(\mathbf{n}_{ab} \times \mathbf{S}_a)(\mathbf{P}_b \cdot \mathbf{n}_{ab}) - ((\mathbf{P}_a \times \mathbf{S}_a)\cdot\mathbf{n}_{ab}) \frac{\mathbf{z}_a + \mathbf{z}_b}{r_{ab}} \right]$$

$$\mathbf{G}_{\mathrm{SS}}^{\mathrm{NLO}} = \frac{1}{2}\sum_a \sum_{b\neq a} \left\{ [3(\mathbf{S}_a\cdot\mathbf{n}_{ab})(\mathbf{S}_b\cdot\mathbf{n}_{ab}) - (\mathbf{S}_a\cdot\mathbf{S}_b)] \frac{\mathbf{z}_a}{r_{ab}^3} + (\mathbf{S}_b\cdot\mathbf{n}_{ab}) \frac{\mathbf{S}_a}{r_{ab}^2} \right\}$$

since 1558

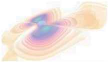


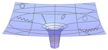


$\Rightarrow$ Poincaré algebra is fulfilled.

Outline

since 1558





Hamiltonians from the Poincaré Algebra

Hergt and Schäfer (2008)

The full Hamiltonian up to 2PN enters the Poincaré algebra:

$$H = H_N + H_{1PN} + H_{2PN} + H_{SO}^{1PN} + H_{SO}^{2PN} + H_{S^2} + H_{S^3 p} + H_{S^2 p^2} + H_{S^4}$$

- Source terms in canonical variables sufficient for $H_{S_2^2 S_1 p_1}$, $H_{S_2^3 p_1}$, $H_{S_1^3 p_2}$, $H_{S_1^2 S_2 p_2}$, $H_{S_1^2 S_2^2}$, $H_{S_1 S_2^3}$, and $H_{S_2 S_1^3}$ were obtained from the Kerr-metric in ADM coordinates (HS 2007).
- Ansatzes for $H_{S_1^2 p^2}$, $H_{S_2^2 p^2}$, $H_{S_1^3 p_1}$, $H_{S_2^3 p_2}$, $H_{S_1^2 S_2 p_1}$, $H_{S_2^2 S_1 p_2}$, $H_{S_1^4}$, and $H_{S_2^4}$ are fixed up to canonical transformation by $\{G_i, H\} = P_i$.
- The static (linear momentum independent) part of the Hamilton constraint is needed to fix these remaining degrees of freedom.

The Stress-Energy Tensor with Quadrupole

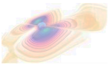
with D. Puetzfeld, arXiv:0909.3756

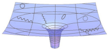
- Stress-energy tensor density with quadrupole has the structure:

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \left[t^{\mu\nu}\delta_{(4)} + (t^{\mu\nu\alpha}\delta_{(4)})_{||\alpha} + (t^{\mu\nu\alpha\beta}\delta_{(4)})_{||\alpha\beta} \right]$$

- Getting expressions for the $t^{\mu\nu\cdots}$ from $T^{\mu\nu}{}_{||\nu} = 0$:
  - Dixon's work: Complicated definitions.
  - Tulczyjew's theorems: Complicated calculation.
- Left to do (unpublished):
  - Relate quadrupole expressions to the quantity $I_1^{ij} \equiv \gamma^{ik}\gamma^{jl}\gamma^{mn}\hat{S}_{1km}\hat{S}_{1nl} + \frac{2}{3}\mathbf{S}_1^2\gamma^{ij}$.
  - Perform the (3+1)-split.
  - Identify canonical variables. . .

since 1558





Ansatz for the Static Source Terms

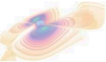
$$\mathscr{H}^{\text{matter}}_{\mathrm{S}_1^2,\,\text{static}} = \frac{c_1}{m_1}\left(I_1^{ij}\delta_1\right)_{;ij} + \frac{c_2}{m_1}\mathrm{R}_{ij}I_1^{ij}\delta_1 + \frac{c_3}{m_1}\mathbf{S}_1^2\left(\gamma^{ij}\delta_1\right)_{;ij} + \frac{c_4}{m_1}\mathrm{R}\mathbf{S}_1^2\delta_1$$
$$+ \frac{1}{8m_1}g_{mn}\gamma^{pj}\gamma^{ql}\gamma^{mi}{}_{,p}\gamma^{nk}{}_{,q}\hat{S}_{1\,ij}\hat{S}_{1\,kl}\delta_1$$
$$+ \frac{1}{4m_1}\left(\gamma^{ij}\gamma^{mn}\gamma^{kl}{}_{,m}\hat{S}_{1\,ln}\hat{S}_{1\,jk}\delta_1\right)_{,i}$$

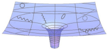
- This ansatz is 3-dim. covariant, as P_i is not:

$$P_i = \int d^3\mathbf{x}\,\mathscr{H}^{\text{matter}}_i = p_i - \frac{1}{2}g_{ij}\gamma^{lm}\gamma^{kj}{}_{,m}\hat{S}_{kl} + \dots$$

- Terms like $I_1^{ij}{}_{;k}\delta_1$ or $I_1^{ij}\delta_{1;k}$ can not appear.
- γ_{ij} for Kerr $\Rightarrow c_1 = -\frac{1}{2}$.
- Lapse function for Kerr $\Rightarrow c_2 = 0$.
- c_3 and c_4 do not contribute to the Hamiltonian.

since 1558

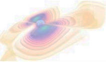


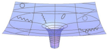


NLO Spin_1-Spin_1 Hamiltonian

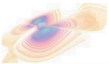
$$
\begin{aligned}
H^{\text{NLO}}_{S_1^2} = \frac{1}{r_{12}^3}\Bigg[& \frac{m_2}{4m_1^3}(\mathbf{P}_1\cdot\mathbf{S}_1)^2 + \frac{3m_2}{8m_1^3}(\mathbf{P}_1\cdot\mathbf{n}_{12})^2\mathbf{S}_1^2 - \frac{3m_2}{8m_1^3}\mathbf{P}_1^2(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 \\
& - \frac{3m_2}{4m_1^3}(\mathbf{P}_1\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{P}_1\cdot\mathbf{S}_1) - \frac{3}{4m_1 m_2}\mathbf{P}_2^2\mathbf{S}_1^2 \\
& + \frac{9}{4m_1 m_2}\mathbf{P}_2^2(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 + \frac{3}{4m_1^2}(\mathbf{P}_1\cdot\mathbf{P}_2)\mathbf{S}_1^2 \\
& - \frac{9}{4m_1^2}(\mathbf{P}_1\cdot\mathbf{P}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 + \frac{3}{4m_1^2}(\mathbf{P}_1\cdot\mathbf{n}_{12})(\mathbf{P}_2\cdot\mathbf{n}_{12})\mathbf{S}_1^2 \\
& - \frac{3}{2m_1^2}(\mathbf{P}_1\cdot\mathbf{n}_{12})(\mathbf{P}_2\cdot\mathbf{S}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12}) \\
& + \frac{3}{m_1^2}(\mathbf{P}_2\cdot\mathbf{n}_{12})(\mathbf{P}_1\cdot\mathbf{S}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12}) \\
& - \frac{15}{4m_1^2}(\mathbf{P}_1\cdot\mathbf{n}_{12})(\mathbf{P}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 \Bigg] \\
- \frac{m_2}{r_{12}^4}\Bigg[& \frac{9}{2}(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 - \frac{5}{2}\mathbf{S}_1^2 + \frac{7m_2}{m_1}(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 - \frac{3m_2}{m_1}\mathbf{S}_1^2 \Bigg]
\end{aligned}
$$

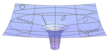
since 1558





Outline





Higher Orders from the Stress-Enery Tensor

with H. Wang, arXiv:0910.1008

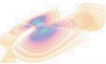
- Need spin corrections to canonical field momentum:

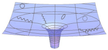
$$\pi^{ij}_{\text{can}} = \pi^{ij}_{\text{field}} + \pi^{ij}_{\text{spin}} ,$$
$$\pi^{ij}_{\text{field}} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{kl} .$$

- Choose π^{ij}_{spin} such that:

$$P_i = \sum_a P_{ai} - \frac{1}{16\pi}\int d^3x\, \pi^{kl\text{TT}}_{\text{can}} h^{\text{TT}}_{kl,i}$$
$$J_{ij} = \sum_a (\hat{z}^i_a P_{aj} - \hat{z}^j_a P_{ai}) + \sum_a \hat{S}_{a(i)(j)}$$
$$- \frac{1}{16\pi}\int d^3x\, (x^i \pi^{kl\text{TT}}_{\text{can}} h^{\text{TT}}_{kl,j} - x^j \pi^{kl\text{TT}}_{\text{can}} h^{\text{TT}}_{kl,i})$$
$$- 2\frac{1}{16\pi}\int d^3x\, (\pi^{ik\text{TT}}_{\text{can}} h^{\text{TT}}_{kj} - \pi^{jk\text{TT}}_{\text{can}} h^{\text{TT}}_{ki})$$

- Got Hamiltonian for field evolution at formal 3.5PN.
- Checked 1PN energy flux (Kidder 1995).





Action Approach

Steinhoff and Schäfer (2009)

- Action:

$$W[e_{a\mu}, z^{\mu}, p_{\mu}, \Lambda^{Ca}, S_{ab}, \lambda_1^a, \lambda_{2[i]}, \lambda_3] = \int d^4x\, \mathscr{L}$$

$$\Lambda^{Aa}\Lambda^{Bb}\eta_{AB} = \eta^{ab}$$

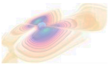
- Lagrangian densities $\mathscr{L} = \mathscr{L}_G + \mathscr{L}_M + \mathscr{L}_C$:

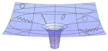
$$\mathscr{L}_M = \int d\tau \left[\left(p_\mu - \frac{1}{2} S_{ab}\, \omega_\mu{}^{ab}\right)\frac{dz^\mu}{d\tau} + \frac{1}{2} S_{ab} \frac{d\theta^{ab}}{d\tau}\right] \delta_{(4)}$$

$$\mathscr{L}_C = \int d\tau \left[\lambda_1^a S_{ab} p^b + \lambda_{2[i]} \Lambda^{[i]a} p_a - \frac{\lambda_3}{2}(p^2 + m^2)\right] \delta_{(4)}$$

$$\mathscr{L}_G = \frac{1}{16\pi}\sqrt{-g}\mathrm{R}^{(4)}$$

$$d\theta^{ab} = \Lambda_C{}^a d\Lambda^{Cb}$$





Reduction of the Matter Part

in covariant SSC and time gauge

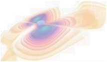
$$\mathscr{L}_M = \mathscr{L}_{MK} + \mathscr{L}_{MC} + \mathscr{L}_{GK}$$

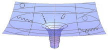
- Kinetic matter part:

$$\mathscr{L}_{MK} = \left[p_i + K_{ij} n S^j + A^{kl} e_{(j)k} e^{(j)}_{l,i} - \left(\frac{1}{2} S_{kj} + \frac{p_{(k} n S_{j)}}{np} \right) \Gamma^{kj}{}_i \right] \dot{z}^i \delta$$
$$+ \frac{n S^i}{2np} \dot{p}_i \delta + \left[S_{(i)(j)} + \frac{n S_{(i)} p_{(j)} - n S_{(j)} p_{(i)}}{np} \right] \frac{\Lambda^{(i)}_{[k]} \dot{\Lambda}^{[k](j)}}{2} \delta$$

- Constraint part $\mathscr{L}_{MC} = N \mathscr{H}^{\text{matter}} - N^i \mathscr{H}_i^{\text{matter}}$:

$$\mathscr{H}^{\text{matter}} = \sqrt{\gamma} T_{\mu\nu} n^\mu n^\nu \,, \qquad \mathscr{H}_i^{\text{matter}} = -\sqrt{\gamma} T_{i\nu} n^\nu$$





Reduction of the Field Part and Result

- Transition to (generalized) Newton-Wigner variables.

$$\Rightarrow \quad \hat{\mathscr{L}}_{GK} = \hat{A}^{ij} e_{(k)i} e_{(k)j,0} \delta \,, \quad g_{ik} g_{jl} \hat{A}^{kl} = \frac{1}{2} \hat{S}_{ij} + \frac{m p_{(i} n S_{j)}}{np(m - np)}$$

- Spatial symmetric gauge (Kibble 1963): $e_{(i)j} = e_{ij} = e_{ji}$

$$e_{ij} e_{jk} = g_{ik} \quad \Rightarrow \quad e_{ij} = \sqrt{g_{ij}}$$

- Definition of field momentum:

$$\pi^{ij}_{\text{can}} = \pi^{ij} + 8\pi \hat{A}^{(ij)} \delta + 16\pi B^{ij}_{kl} \hat{A}^{[kl]} \delta$$

$$e_{k[i} e_{j]k,0} = B^{kl}_{ij} g_{kl,0}$$

- Result:

$$W = \frac{1}{16\pi} \int d^4 x \, \pi^{ij\mathrm{TT}}_{\text{can}} h^{\mathrm{TT}}_{ij,0} + \int dt \left[P_i \dot{\hat{z}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \frac{d\hat{\theta}^{(i)(j)}}{dt} - E \right]$$

Thank you for your attention!